Bi-Laplacians on graphs and networks

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joint work with Delio Mugnolo

Motivation

Our aim is to study

$$u_t(t,x) = -\Delta^2 u(t,x) \qquad \text{on some graph.}$$

Modelling aspects:

- behaviour of flexible structures;
- non linear elasticity;
- the solution of $u_t = -\Delta^2 u$ is the expected value of random solutions (normally distributed in time) of $u_t = i\Delta u$ (Griego-Hersch 1969).

Motivation

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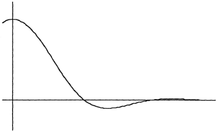
$$u_t(t,x) = -\Delta^2 u(t,x) \qquad \text{on some graph.}$$

Interesting mathematical features:

- no maximum principles;
- no positivity preserving properties:
  $e^{-t\Delta^m} \not\geq 0$ for any $m > 1$; for $m = 2$ the heat kernel
  $p(t,x) = t^{-\frac{1}{4}} p(1, t^{-\frac{1}{4}} x)$ with

  $p(1,\cdot) =$ 

  (Davies 1995)
- No relation between Sobolev inequality and L^∞-contractivity.

History

Davies in 1995 studied

$$\begin{cases} u_t(t,x) = -\Delta^2 u(t,x) & t \geq 0, x \in \mathbb{R} \\ u(0,\cdot) = f(\cdot) \in L^2(\mathbb{R}). \end{cases}$$

→ bounded semigroup $T(\cdot)$ on $L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$ given by

$$T(t)f(x) = \int_{-\infty}^{\infty} K(t,x,y)f(y)\,dy, \quad f \in L^p(\mathbb{R}),$$

and for all $t > 0$, $x, y \in \mathbb{R}$ and some $c_1, c_2, k > 0$

$$|K(t,x,y)| \leq c_1 t^{-1/4} e^{-c_2 \frac{|x-y|^{4/3}}{t^{1/3}} + kt}.$$

→ $T(t)$ is L^1-uniformly bounded → C_0-semigroup $T(\cdot)$ on $L^p(\mathbb{R})$ for all $1 \leq p < \infty$.

History

Theorem 1 (Gazzola-Grunau 2008)

If

- $f \geq 0$, $f \neq 0$,
- f *is continuous and compactly supported,*

then the solution to

$$\begin{cases} u_t(t,x) = -\Delta^2 u(t,x) & \mathit{in}\ [0,\infty) \times \mathbb{R}, \\ u(0,x) = f(x) & \mathit{in}\ \mathbb{R}, \end{cases}$$

is **eventually positive**, *i.e., for any* $I \subset \mathbb{R}$ *there exists a* $T_I = T_I(f) > 0$ *such that* $u(t,x) > 0$ *for all* $t \geq T_I$ *and* $x \in I$*; and there esists* $\tau = \tau(f) > 0$ *such that for all* $t > \tau$ *there exists a* $x_t \in \mathbb{R}$ *such that* $u(t, x_t) < 0$.

Setting

- *DISCRETE GRAPH*
  Consider a (finite or infinite but *uniformly locally finite*) graph
  $\mathsf{G} = (\mathsf{V}, \mathsf{E})$,
  with V vertices and E edges (i.e., $V = |\mathsf{V}|$ and $E = |\mathsf{E}|$).
  Assume G to have neither loops nor multiple edges.
- *METRIC GRAPH*
  Assume G finite and connected and $0 < \deg(\mathsf{v}) < \infty$, $\forall \mathsf{v} \in \mathsf{V}$.
  Identify each edge $\mathsf{e} \equiv (\mathsf{v}, \mathsf{w})$ with $[0, \ell_{\mathsf{e}}]$ and its endpoints v, w with 0 and ℓ_{e}.
  Denote by $\mathcal{G}$ the resulting metric measure space.

Semigroups in short

$T : [0, \infty) \to \mathcal{L}(X)$, X Banach space.
Operator family $(T(t))_{t \geq 0}$ is called a strongly continuous semigroup if

- $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$;
- $t \mapsto T(t)x \in X$ is continuous for every $x \in X$.

Semigroups in short

The generator of a strongly continuous semigroup A is defined as

$$D(A) := \{x \in X : t \mapsto T(t)x \text{ is differentiable on } [0,\infty)\}$$

$$Ax := \frac{d}{dt} T(t)x_{|t=0} = \lim_{t \downarrow 0} \frac{1}{t}(T(t)x - x).$$

$$T(t) \rightsquigarrow e^{tA}$$

Semigroup approach to initial-boundary value problems

Given X Banach space, $A : D(A) \subset X \to X$ with boundary conditions in $D(A)$

$$(ACP_1) \begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0 \end{cases} \quad \text{and} \quad (ACP_2) \begin{cases} \ddot{x}(t) = Ax(t) \\ x(0) = x_0 \\ \dot{x}(0) = x_1 \end{cases}$$

- $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t \geq 0}$ on $X \Leftrightarrow (ACP_1)$ well posed with solution $x(t) = T(t)x_0$.
- Study qualitative properties: positivity, stability, regularity, ...
- (ACP_2) well posed $\Leftrightarrow (A, D(A))$ generates a cosine family on X.

Discrete bi-Laplacian

Let G be a finite graph or infinite but uniformly locally finite graph.

-$\mathcal{L} \rightarrow$ generates a semigroup on the Hilbert space $\ell^2(V)$. In particular, $\mathcal{L}$ is a (bounded), positive semidefinite self-adjoint operator, hence

$\mathcal{L}^2$ is a (bounded), positive semidefinite self-adjoint operator on $\ell^2(V)$.
$\Rightarrow -\mathcal{L}^2$ generates a cosine operator function and an analytic semigroup of angle $\frac{\pi}{2}$ on $\ell^2(V)$.

Proposition 1

Let G be a connected graph. Then the following assertions are equivalent.

- G is complete.
- $(e^{-t\mathcal{L}^2})_{t\geq 0}$ is positive.
- $(e^{-t\mathcal{L}^2})_{t\geq 0}$ is ℓ^∞-contractive.

Weaker contractivity properties for all graphs.

Proposition 2

The semigroup $(e^{-t\mathcal{L}^2})_{t\geq 0}$ is ℓ^p-contractive for some $p \in (2, \infty)$.

bi-Laplacian on network

We identify functions on $\mathcal{G}$ as vectors $(u_e)_{e\in E}$, where each u_e is defined on the edge $e \simeq (0,\ell_e)$ and introduce

$$L^2(\mathcal{G}) = \bigoplus_{e\in E} L^2(0,\ell_e)$$
$$= \left\{ (u_e)_{e\in E} \text{ s.t. } u_e : (0,\ell_e) \to \mathbb{C} \text{ is meas. and } \sum_{e\in E} \int_0^{\ell_e} |u_e(x)|^2\, dx < \infty \right\},$$

and, we will denote for every $k \in \mathbb{N}$ by

$$\widetilde{H}^k(\mathcal{G}) = \bigoplus_{e\in E} H^k(0,\ell_e)$$

the space of H^k functions on the (open) edges where we will specify later the vertex conditions.

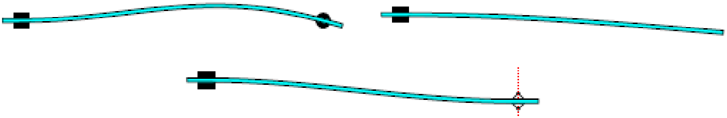
The operator

We want to study on $\mathcal{G}$ the differential operator

$$A = \Delta^2 = \frac{d^4}{dx^4}$$

with domain

$$D(A) = \{u \,:\, u_{\mathsf{e}} \in H^4(0, \ell_{\mathsf{e}}) \text{ for each } \mathsf{e} \in \mathsf{E} + \text{vertex conditions involving the function and its derivatives at nodes which make the operator self-adjoint}\}.$$



(Gazzola-Grunau-Sweers)

- CLAMPED $u(0) = u'(0) = 0$
- HINGED $u(1) = u''(1) = 0$
- FREE $u''(1) = u'''(1) = 0$
- FREE VERTICAL SLIDING but with fixed derivative $u'(1) = u'''(1) = 0$

Recall: If A_0 is a symmetric, positive semi-definite operator $\Rightarrow$ it has self-adjoint extentions.
They can be parametrized (Friedrichs theory) and ordered wrt:

$$A_1 \leq A_2 \Leftrightarrow D(a_2) \subset D(a_1) \text{ and } a_1(x) \leq a_2(x) \ \forall \ x \in D(a_2).$$

- NOT a total order;
- BUT $\exists$ one largest extension (Friedrichs),
  and one smallest extension (Krein-von Neumann).

Integrating by parts

$$
\begin{aligned}
(Au, v)_{L^2(\mathcal{G})} &= \sum_{e\in E} \int_0^{\ell_e} u_e''''(x)\overline{v_e(x)}\,dx \\
&= \sum_{e\in E} \left[u_e'''\overline{v_e}\right]_0^{\ell_e} - \sum_{e\in E} \left[u_e''\overline{v_e'}\right]_0^{\ell_e} + \sum_{e\in E} \left[u_e'\overline{v_e''}\right]_0^{\ell_e} \\
&\quad - \sum_{e\in E} \left[u_e\overline{v_e'''}\right]_0^{\ell_e} + \sum_{e\in E} \int_0^{\ell_e} u_e(x)\overline{v_e''''(x)}\,dx.
\end{aligned}
$$

Integrating by parts

$$(Au, v)_{L^2(\mathcal{G})} = \sum_{e\in E} \int_0^{\ell_e} u_e''''(x)\overline{v_e(x)}\, dx$$

$$= \sum_{e\in E} \left[u_e'''\overline{v_e}\right]_0^{\ell_e} - \sum_{e\in E} \left[u_e''\overline{v_e'}\right]_0^{\ell_e} + \sum_{e\in E} \left[u_e'\overline{v_e''}\right]_0^{\ell_e}$$

$$- \sum_{e\in E} \left[u_e\overline{v_e'''}\right]_0^{\ell_e} + \sum_{e\in E} \int_0^{\ell_e} u_e(x)\overline{v_e''''(x)}\, dx.$$

Denoting as $u(0) := (u_{\mathsf{e}}(0))_{\mathsf{e}\in\mathsf{E}}$ and $u(\ell) := (u_{\mathsf{e}}(\ell_{\mathsf{e}}))_{\mathsf{e}\in\mathsf{E}}$, the boundary equality means that

$$\left(\begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix}, \begin{pmatrix} -v'''(0) \\ v'''(\ell) \\ -v''(0) \\ -v''(\ell) \end{pmatrix} \right)_{\mathbb{C}^{4E}} = \left(\begin{pmatrix} -u'''(0) \\ u'''(\ell) \\ -u''(0) \\ -u''(\ell) \end{pmatrix}, \begin{pmatrix} v(0) \\ v(\ell) \\ -v'(0) \\ v'(\ell) \end{pmatrix} \right)_{\mathbb{C}^{4E}}$$

for all $u, v \in D(A)$.

Let Y be a subspace of $\mathbb{C}^{4E}$, then this condition reads as

$$\begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix}, \begin{pmatrix} v(0) \\ v(\ell) \\ -v'(0) \\ v'(\ell) \end{pmatrix} \in Y \text{ and } \begin{pmatrix} -u'''(0) \\ u'''(\ell) \\ -u''(0) \\ -u''(\ell) \end{pmatrix}, \begin{pmatrix} -v'''(0) \\ v'''(\ell) \\ -v''(0) \\ -v''(\ell) \end{pmatrix} \in Y^\perp .$$

However, this boundary condition can be generalised considering $R \in \mathcal{L}(Y)$ and imposing

$$\begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix} \in Y, \qquad \begin{pmatrix} -u'''(0) \\ u'''(\ell) \\ -u''(0) \\ -u''(\ell) \end{pmatrix} + R \begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix} \in Y^\perp$$

on all u in the domain of A.

Theorem 2

Let $A = \frac{d^4}{dx^4}$ *acting on each edge of* $\mathcal{G}$. *Let* $D(A) = \{u \,:\, u_e \in H^4(0,\ell_e)\,\forall\, e + \textit{vertex cond.}\}$. *Then, TFAE:*

(i) *A is self-adjoint;*

(ii) *the vertex conditions can be written as*

$$\begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix} \in Y, \qquad \begin{pmatrix} -u'''(0) \\ u'''(\ell) \\ -u''(0) \\ -u''(\ell) \end{pmatrix} + R \begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix} \in Y^{\perp}$$

where Y *is a subspace of* $\mathbb{C}^{4E}$ *and* $R \in \mathcal{L}(Y)$ *is self-adjoint;*

(iii) *the vertex conditions can be written as*

$$C \begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix} + B \begin{pmatrix} -u'''(0) \\ u'''(\ell) \\ -u''(0) \\ -u''(\ell) \end{pmatrix} = 0$$

for suitable $C, B \in M^{4E}(\mathbb{C})$ *s. t.* (CB) *has max rank and* CB^* *is self-adj.*

Some conditions in literature

Dekoninck and Nicaise

B. Dekoninck, S. Nicaise, *Control of networks of Euler-Bernouilli beams*, ESAIM Control oprim. Calc. Var. **4** (1999), 57-81.

study the exact controllability problem of hyperbolic systems of networks of beams with the following conditions (adapted to our notations)

$$\begin{cases} u_e(v) = u_f(v) & \text{if } e \cap f = v, \\ \sum_{e \in E_v} \frac{\partial u_e}{\partial \nu}(v) = 0 & \forall\, v \in V, \\ u_e''(v) = u_f''(v) & \text{if } e \cap f = v, \\ \sum_{e \in E_v} \frac{\partial^3 u_e}{\partial \nu^3}(v) = 0 & \forall\, v \in V, \end{cases} \tag{1}$$

where $(\partial u_e)/(\partial \nu)(v)$ is the exterior normal derivative of u_e at v.

The same authors

B. Dekoninck, S. Nicaise, *The eigenvalue problem for networks on beams*, Linear Algebra Appl. **314** (2000), 165-189.

study the characteristic equation for the spectrum of the operator with the following two sets of conditions

$$\begin{cases} u_e(v) = u_f(v) & \text{if } e \cap f = v, \\ \frac{\partial u_e}{\partial \nu}(v) = \frac{\partial u_f}{\partial \nu}(v) & \text{if } e \cap f = v, \\ \sum_{e \in E_v} u_e''(v) = 0 & \forall\ v \in V, \\ \sum_{e \in E_v} \frac{\partial^3 u_e}{\partial \nu^3}(v) = 0 & \forall\ v \in V. \end{cases} \tag{2}$$

and

$$\begin{cases} u_{\mathsf{e}}(\mathsf{v}) = u_{\mathsf{f}}(\mathsf{v}) & \text{if } \mathsf{e} \cap \mathsf{f} = \mathsf{v}, \\ u''(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}, \\ \sum_{\mathsf{e} \in \mathsf{E}_{\mathsf{v}}} \frac{\partial^3 u_{\mathsf{e}}}{\partial \nu^3}(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}. \end{cases} \tag{3}$$

Consider the restriction of A to the space

$$D_{cont}(A) := \{f \in C(\mathcal{G}) : f' \in \bigoplus C_c^\infty(0, \ell_\mathsf{e})\}.$$

- *FRIEDRICHS EXTENSION A_F*

$$\begin{cases} u_\mathsf{e}(\mathsf{v}) = u_\mathsf{f}(\mathsf{v}) & \text{if } \mathsf{e} \cap \mathsf{f} = \mathsf{v}, \\ \frac{\partial u_\mathsf{e}}{\partial \nu}(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}, \\ \sum_{\mathsf{e} \in \mathsf{E}_\mathsf{v}} \frac{\partial^3 u_\mathsf{e}}{\partial \nu^3}(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}. \end{cases} \tag{4}$$

- *KREIN-von NEUMANN EXTENSION A_K*

$$\begin{cases} u_\mathsf{e}(\mathsf{v}) = u_\mathsf{f}(\mathsf{v}) & \text{if } \mathsf{e} \cap \mathsf{f} = \mathsf{v}, \\ u_\mathsf{e}''(0) = u_\mathsf{e}''(\ell_\mathsf{e}) = \frac{u_\mathsf{e}'(\ell_\mathsf{e}) - u_\mathsf{e}'(0)}{\ell_\mathsf{e}} & \\ \sum_{\mathsf{e} \in \mathsf{E}_\mathsf{v}} \frac{\partial^3 u_\mathsf{e}}{\partial \nu^3}(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}. \end{cases} \tag{5}$$

The semigroup

The operator

$$Au = \frac{d^4 u}{dx^4},$$

$$D(A) = \left\{ u \in \widetilde{H}^4(\mathcal{G}) : \begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix} \in Y \text{ and } \begin{pmatrix} -u'''(0) \\ u'''(\ell) \\ -u''(0) \\ -u''(\ell) \end{pmatrix} + R \begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix} \in Y^\perp \right\}$$

with Y subspace of $\mathbb{C}^{4E}$ and R self-adjoint operator on Y, is self-adjoint.

The quadratic form associated with A is given by

$$\mathfrak{a}(u) = \sum_{\mathsf{e}\in\mathsf{E}} \int_0^{\ell_\mathsf{e}} |u_\mathsf{e}''(x)|^2 \, dx - \left(R \begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix}, \begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix} \right)$$

with domain

$$D(\mathfrak{a}) := \widetilde{H}^2_Y(\mathcal{G}) := \left\{ u \in \widetilde{H}^2(\mathcal{G}) : \begin{pmatrix} u(0) \\ u(\ell) \\ -u'(0) \\ u'(\ell) \end{pmatrix} \in Y \right\}.$$

Let Y be a subspace of $\mathbb{C}^{4E}$ and $R \in \mathcal{L}(Y)$.

- *PROPERTIES OF THE FORM*
  $\mathfrak{a}$ is densely defined, continuous, $L^2(\mathcal{G})$-elliptic and of Lions type:
- *THEN*
  $-A$ generates a C_0-semigroup $(e^{-tA})_{t\geq 0}$ in $L^2(\mathcal{G})$ that is analytic of angle $\frac{\pi}{2}$.
  e^{-tA} is of trace class (and in particular compact) for all $t > 0$.
  e^{-tA} is self-adjoint if and only if R is self-adjoint;
  e^{-tA} is contractive if R is dissipative.

Proposition 3

Let Y and R be as above and R dissipative. Then the semigroup $(e^{-tA})_{t\geq 0}$ is ultracontractive; in particular, it has an integral kernel of class $L^\infty(\mathcal{G}\times\mathcal{G})$ and the estimate

$$\|e^{-tA}\|_{1\to\infty} \leq ct^{-\frac{1}{4}}e^{\frac{t}{2}} \quad \text{for all } t>0 \tag{6}$$

holds.

CONSEQUENCE

$(e^{-tA})_{t\geq 0}$ on $L^2(\mathcal{G})$ extrapolates to a consistent family of semigroups on $L^p(\mathcal{G})$ for every $1\leq p\leq\infty$.

Eventual positivity

Definition 3

Let $(e^{-tL})_{t\geq 0}$ be a real C_0-semigroup on a Banach space X.

- The semigroup $(e^{-tL})_{t\geq 0}$ is called *uniformly eventually (strongly) positive* if there exists $t_0 \geq 0$ such that $e^{-tL} \geq 0$ ($\gg 0$) for all $t \geq t_0$.

D. Daners, J. Glück, and J.B. Kennedy, *Eventually positive semigroups of linear operators.* J. Math. Anal. Appl., **433**:1561-1593, 2016.

J. Glück, *Invariant sets and long time behaviour of operator semigroups*, Ph.D. Thesis, University of Ulm, 2016.

Characterization to prove eventual positivity of the semigroup through the spectrum of the operator.

Proposition 4

Let X be a σ-finite measure space and $(T(t))_{t\geq 0}$ be a real, strongly continuous semigroup on the complex Hilbert lattice $L^2(X)$ with self-adjoint generator $-A$. Let $u > 0$ a.e. and assume that $\bigcap_{k=1}^{\infty} D(A^k) \subset H_u$.
Then $(T(t))_{t\geq 0}$ is uniformly eventually strongly positive with respect to u if and only if the spectral bound $s(-A)$ is a simple eigenvalue and the associated eigenspace contains a vector v such that $v \gg_u 0$.

The operator A is

- positive
- self-adjoint
- with compact resolvent

Then A has positive, real, pure point spectrum
$\Rightarrow s(-A) = 0$ as long as $\mathbf{1} \in D(A)$.
And in this case if $R = 0$

$$u_e(x) = a_e x + b_e \qquad \forall\, e \in E.$$

If $R \neq 0$

$$u_e(x) = a_e x^3 + b_e x^2 + c_e x + d_e \qquad \forall\, e \in E.$$

Examples

Proposition 5

The semigroup generated by the Friedrichs extension $-A_F$ of the bi-Laplacian $(A, D_{cont}(A))$ is eventually positive. The semigroup generated by the Krein–von Neumann extension $-A_K$ of the bi-Laplacian $(A, D_{cont}(A))$ is not eventually positive.

$$\begin{cases} u_e(v) = u_f(v) & \text{if } e \cap f = v, \\ \frac{\partial u_e}{\partial \nu}(v) = 0 & \forall\, v \in V, \\ \sum_{e \in E_v} \frac{\partial^3 u_e}{\partial \nu^3}(v) = 0 & \forall\, v \in V. \end{cases} \quad \begin{cases} u_e(v) = u_f(v) & \text{if } e \cap f = v, \\ u_e''(0) = u_e''(\ell_e) = \frac{u_e'(\ell_e) - u_e'(0)}{\ell_e} & \\ \sum_{e \in E_v} \frac{\partial^3 u_e}{\partial \nu^3}(v) = 0 & \forall\, v \in V. \end{cases}$$

Proposition 6

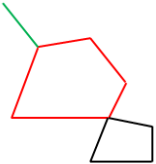
Let $\mathcal{G}$ be a tree. Then the semigroup generated by the operator $-A$ with conditions (1) in all transmission vertices (i.e., on all vertices of degree ≥ 2) is uniformly eventually positive if and only if Neumann boundary conditions are imposed on all leaves (i.e., on all vertices of degree 1), up to at most one exception.

$$\begin{cases} u_{\mathsf{e}}(\mathsf{v}) = u_{\mathsf{f}}(\mathsf{v}) & \text{if } \mathsf{e} \cap \mathsf{f} = \mathsf{v}, \\ \sum_{\mathsf{e} \in \mathsf{E}_{\mathsf{v}}} \frac{\partial u_{\mathsf{e}}}{\partial \nu}(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}, \\ u''_{\mathsf{e}}(\mathsf{v}) = u''_{\mathsf{f}}(\mathsf{v}) & \text{if } \mathsf{e} \cap \mathsf{f} = \mathsf{v}, \\ \sum_{\mathsf{e} \in \mathsf{E}_{\mathsf{v}}} \frac{\partial^3 u_{\mathsf{e}}}{\partial \nu^3}(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}, \end{cases}$$

Proposition 7

Let $\mathcal{G}$ be a graph containing at least one *odd* cycle. Then the semigroup generated by the operator $-A$ with conditions (2) in all transmission vertices is uniformly eventually positive if Dirichlet boundary conditions are not imposed on any vertex of degree 1 (if such vertices exist at all).

$$\begin{cases} u_e(v) = u_f(v) & \text{if } e \cap f = v, \\ \frac{\partial u_e}{\partial \nu}(v) = \frac{\partial u_f}{\partial \nu}(v) & \text{if } e \cap f = v, \\ \sum_{e \in E_v} u_e''(v) = 0 & \forall\, v \in V, \\ \sum_{e \in E_v} \frac{\partial^3 u_e}{\partial \nu^3}(v) = 0 & \forall\, v \in V. \end{cases}$$



Let A be endowed with conditions

$$\begin{cases} u_{\mathsf{e}}(\mathsf{v}) = u_{\mathsf{f}}(\mathsf{v}) & \text{if } \mathsf{e} \cap \mathsf{f} = \mathsf{v}, \\ u''(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}, \\ \sum_{\mathsf{e} \in \mathsf{E}_{\mathsf{v}}} \frac{\partial^3 u_{\mathsf{e}}}{\partial \nu^3}(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}. \end{cases}$$

We do not obtain eventually positivity because only continuity on the trace but no conditions on the first derivative is imposed in the vertices, then it is clear that *all* polynomials of degree 1 (with suitable coefficients of degree 0 realizing the continuity condition) lie in the null space.

Eventual L^∞-contractivity

Proposition 8

Let (Ω, μ) be a finite measure space and $-A$ a self-adjoint operator with compact resolvent on $L^2(\Omega)$. Let 0 be the spectral bound of $-A$ and let the associated eigenspace E_0 be a one-dimensional space spanned by $\mathbf{1}$. Also assume that $\bigcap_{k\in\mathbb{N}} D(A^k) \hookrightarrow L^\infty(\Omega)$ and e^{-tA} be real. Then the semigroup generated by $-A$ is eventually Markovian, i.e., eventually positive and eventually L^∞-contractive.

Proposition 9

The semigroup generated by the bilaplacian operator endowed with conditions

$$\begin{cases} u_{\mathsf{e}}(\mathsf{v}) = u_{\mathsf{f}}(\mathsf{v}) & \text{if } \mathsf{e} \cap \mathsf{f} = \mathsf{v}, \\ \frac{\partial u_{\mathsf{e}}}{\partial \nu}(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}, \\ \sum_{\mathsf{e} \in \mathsf{E}_{\mathsf{v}}} \frac{\partial^3 u_{\mathsf{e}}}{\partial \nu^3}(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}. \end{cases} \quad \text{or} \quad \begin{cases} u_{\mathsf{e}}(\mathsf{v}) = u_{\mathsf{f}}(\mathsf{v}) & \text{if } \mathsf{e} \cap \mathsf{f} = \mathsf{v}, \\ \sum_{\mathsf{e} \in \mathsf{E}_{\mathsf{v}}} \frac{\partial u_{\mathsf{e}}}{\partial \nu}(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}, \\ u''_{\mathsf{e}}(\mathsf{v}) = u''_{\mathsf{f}}(\mathsf{v}) & \text{if } \mathsf{e} \cap \mathsf{f} = \mathsf{v}, \\ \sum_{\mathsf{e} \in \mathsf{E}_{\mathsf{v}}} \frac{\partial^3 u_{\mathsf{e}}}{\partial \nu^3}(\mathsf{v}) = 0 & \forall\ \mathsf{v} \in \mathsf{V}, \end{cases}$$

is eventually Markovian.

Proposition 10

Let G be finite. Then the semigroup $(e^{-t\mathcal{L}^2})_{t\geq 0}$ is uniformly eventually positive and eventually ℓ^∞-contractive, hence eventually Markovian.

Thank you for your attention!