Time asymptotics of structured populations with diffusion and Feller boundary conditions

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01/10/2018
Conference "Semigroups of Operators: Theory and Applications"

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Size-structured population model

We consider the model: Farkas and Hinow (2011)

$$\partial_t u(t,s) + \overbrace{\partial_s(\gamma(s)u(t,s))}^{transport} = \overbrace{\partial_s(d(s)\partial_s(u(t,s)))}^{diffusion} - \overbrace{\mu(s)u(t,s)}^{mortality} + \underbrace{\int_0^m \beta(s,y)u(t,y)\mathrm{d}y}_{reproduction}, \quad \forall s \in [0,m],$$

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with Feller boundary conditions

$$[\partial_s(d(s)\partial_s u(t,s))]_{s=0} - b_0\partial_s u(t,0) + c_0 u(t,0) = 0,$$
$$[\partial_s(d(s)\partial_s u(t,s))]_{s=m} + b_m\partial_s u(t,m) + c_m u(t,m) = 0.$$

$\rightarrow$ Asymptotic behavior of the solutions ?

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Hypotheses

1. The functions μ, γ' and $s \mapsto \beta(s, y)$ are continuous at $s = 0$ and at $s = m$ for every $y \in [0, m]$;
2. Let $\beta_0 = \beta(0, .)$ and $\beta_m = \beta(m, .)$;

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2. Let $\beta_0 = \beta(0, .)$ and $\beta_m = \beta(m, .)$;
3. $\gamma, d \in W^{1,\infty}(0, m)$ and $\mu, \beta_0, \beta_m \in L^\infty(0, m)$;
4. $b_0, b_m > 0,\ c_0, c_m \geq 0,\ \beta, \mu \geq 0$ and $d(s) \geq d_0 > 0$ for every $s \in [0, m]$;

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4. $b_0, b_m > 0,\ c_0, c_m \geq 0,\ \beta, \mu \geq 0$ and $d(s) \geq d_0 > 0$ for every $s \in [0,m]$;
5. the operator

$$L^1(0,m) \ni u \to \int_0^m \beta(\cdot, y) u(y) \mathrm{d}y \in L^1(0,m)$$

is weakly compact.

Dynamic boundary conditions

We rewrite the boundary conditions under the dynamics form

$$\partial_t u(t,0) = -u(t,0)\rho_0 + \partial_s u(t,0)(b_0 - \gamma(0)) + \int_0^m \beta_0(y)u(t,y)\mathrm{d}y,$$
$$\partial_t u(t,m) = -u(t,m)\rho_m - \partial_s u(t,m)(b_m + \gamma(m)) + \int_0^m \beta_m(y)u(t,y)\mathrm{d}y,$$

where

$$\rho_0 = \gamma'(0) + \mu(0) + c_0, \quad \rho_m = \gamma'(m) + \mu(m) + c_m.$$

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$$\rho_0 = \gamma'(0) + \mu(0) + c_0, \quad \rho_m = \gamma'(m) + \mu(m) + c_m.$$

We work in the Banach space

$$\mathcal{X} = (L^1(0,m) \times \mathbb{R}^2, \|.\|_{\mathcal{X}}),$$

$$\|(u, u_0, u_m)\|_{\mathcal{X}} = \|u\|_{L^1(0,m)} + c_1|u_0| + c_2|u_m|,$$

$$\text{where} \quad c_1 = \frac{d(0)}{b_0 - \gamma(0)}, \quad c_2 = \frac{d(m)}{b_m + \gamma(m)}.$$

Well-posedness

Let

$$
\begin{aligned}
A_K \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} &= A \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} + K \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} \\
&= \begin{pmatrix} (du')' - (\gamma u)' - \mu u \\ (b_0 - \gamma(0))u'(0) - \rho_0 u_0 \\ -(b_m + \gamma(m))u'(m) - \rho_m u_m \end{pmatrix} + \begin{pmatrix} \int_0^m \beta(\cdot, y)u(y)\mathrm{d}y \\ \int_0^m \beta_0(y)u(y)\mathrm{d}y \\ \int_0^m \beta_m(y)u(y)\mathrm{d}y \end{pmatrix},
\end{aligned}
$$

in a suitable domain $D(A)$. We get

$$
\begin{cases} U'(t) = A_K U(t), \\ U(0) = (u^0, u_0^0, u_m^0) \in \mathcal{X}, \end{cases}
$$

for $U(t) = (u(t), u_0(t), u_m(t))^T$.

Well-posedness

Let

$$A_s : D(A_s) \to \mathcal{X},$$

$$D(A_s) = \{(u, u_0, u_m) \in C^2[0,m] \times \mathbb{R}^2 : u(0) = u_0, u(m) = u_m\} \subset D(A_K).$$

Farkas and Hinow (2011):

- A_s is dissipative;
- the closure of A_s is a generator.

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- the closure of A_s is a generator.

Theorem

The domain of the generator is

$$D(A_K) = \{(u, u_0, u_m) \in W^{2,1}(0, m) \times \mathbb{R}^2 : u(0) = u_0, u(m) = u_m\}.$$

Asynchronous exponential growth

Definition (Webb (1987))

Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup of bounded linear operators in $\mathcal{X}$. The semigroup has the property of **asynchronous exponential growth** with intrinsic growth constant $\lambda_0 \in \mathbb{R}$ if there exists a nonzero finite rank operator P_0 in $\mathcal{X}$, such that

$$\lim_{t\to\infty} e^{-\lambda_0 t} T(t) = P_0.$$

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In practice, this behavior relies on two conditions:

- the **irreducibility** of the semigroup;
- the existence of a **spectral gap**.

Irreducibility

- A positive C_0-semigroup $\{T(t)\}_{t\geq 0}$ in $\mathcal{X}$ is **irreducible** if, for every $f \in \mathcal{X}$, $f > 0$ and $x \in \mathcal{X}'$, $x > 0$, there exists $t > 0$ such that $\langle T(t)f, x\rangle > 0$;
- A positive operator $\mathcal{A}$ in $\mathcal{X}$ is **positivity improving** if, for every $f \in \mathcal{X}$, $f > 0$ and $x \in \mathcal{X}'$, $x > 0$, we have $\langle \mathcal{A}f, x\rangle > 0$.

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Let $\{T_{\mathcal{A}}(t)\}_{t\geq 0}$ a positive C_0-semigroup in $\mathcal{X}$, with generator $\mathcal{A}$. Then the semigroup is **irreducible** if and only if, for λ large enough, the resolvent $(\lambda - \mathcal{A})^{-1}$ is **positivity improving**.

Spectral gap

A C_0-semigroup $\{T(t)\}_{t\geq 0}$ has a **spectral gap** if

$$\omega_{\text{ess}}(\{T(t)\}_{t\geq 0}) < \omega_0(\{T(t)\}_{t\geq 0}),$$

where

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Let $\{T(t)\}_{t\geq 0}$ a positive C_0-semigroup in $\mathcal{X}$. If

$$\omega_{\text{ess}}(\{T(t)\}_{t\geq 0}) < \omega_0(\{T(t)\}_{t\geq 0}),$$

and if $\{T(t)\}_{t\geq 0}$ is **irreducible**, then $\{T(t)\}_{t\geq 0}$ has the property of asynchronous exponential growth, with rank one projection operator P_0.

Irreducibility of the semigroup

Under the assumption $\mathcal{C}([0,m]^2) \ni \beta(\cdot,\cdot) > 0$ a.e., the C_0-semigroup $\{T_{A_K}(t)\}_{t\geq 0}$ generated by A_K is irreducible: Farkas and Hinow (2011).

Theorem

The C_0-semigroup $\{T_{A_K}(t)\}_{t\geq 0}$ is irreducible.

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Theorem

The C_0-semigroup $\{T_{A_K}(t)\}_{t\geq 0}$ is irreducible.

Proof.

For λ large enough, we have

$$\begin{aligned}(\lambda I - A_K)^{-1} &= (\lambda I - A)^{-1} + (\lambda I - A)^{-1}\sum_{n=1}^{\infty}(K(\lambda I - A)^{-1})^n \\ &\geq (\lambda - A)^{-1}\end{aligned}$$

and $(\lambda I - A)^{-1}$ is positivity improving (Hopf's maximum principle). □

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Theorem

If $K \neq 0$, then $\{T_{A_K}(t)\}_{t\geq 0}$ has the property of asynchronous exponential growth.

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Proof: the semigroup $\{T_{A_K}(t)\}_{t\geq 0}$ is **irreducible** and we have

$$T_{A_K}(t) = T_A(t) + \int_0^t T_A(t-s) K T_{A_K}(s) ds.$$

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$$T_{A_K}(t) = T_A(t) + \int_0^t T_A(t-s) K T_{A_K}(s) ds.$$

Since K is weakly compact, then the strong integral

$$\int_0^t T_A(t-s) K T_{A_K}(s) ds$$

is a weakly compact operator (Schlüchtermann (1992)).

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is a weakly compact operator (Schlüchtermann (1992)). Moreover, $\{T_A(t)\}_{t\geq 0}$ and $\{T_{A_K}(t)\}_{t\geq 0}$ have the same essential spectrum, and

$$\omega_{\text{ess}}(\{T_{A_K}(t)\}_{t\geq 0}) = \omega_{\text{ess}}(\{T_A(t)\}_{t\geq 0}) \leq \omega_0(\{T_A(t)\}_{t\geq 0}) = s(A).$$

Asynchronous exponential growth

The resolvent $(\lambda - A_K)^{-1}$ is compact, irreducible and

$$(\lambda - A)^{-1} \leq (\lambda - A_K)^{-1}, \qquad 0 \leq (\lambda - A)^{-1} \neq (\lambda - A_K)^{-1}.$$

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Marek's comparison theorem (1970) implies

$$r_\sigma((\lambda - A)^{-1}) < r_\sigma((\lambda - A_K)^{-1}),$$

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so

$$\frac{1}{\lambda - s(A)} = r_\sigma((\lambda - A)^{-1}) < r_\sigma((\lambda - A_K)^{-1}) = \frac{1}{\lambda - s(A_K)}$$

for λ large enough.

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for λ large enough. Consequently

$$\omega_{\text{ess}}(\{T_{A_K}(t)\}_{t \geq 0}) \leq s(A) < s(A_K) = \omega_0(\{T_{A_K}(t)\}_{t \geq 0})$$

and therefore there is a **spectral gap**. □

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Model

In the case $m = \infty$, we study the model

$$\begin{array}{rcl}\partial_t u(t,s) + \partial_s(\gamma(s)u(t,s)) & = & \partial_s(d(s)\partial_s u(t,s)) - \mu(s)u(t,s) \\ & & + \int_0^\infty \beta(s,y)u(y,t)\mathrm{d}y, \\ [\partial_s(d(s)\partial_s u(t,s))]_{s=0} & - & b_0\partial_s u(t,0) + c_0 u(t,0) = 0.\end{array}$$

Same hypotheses as in the finite case, then we rewrite the boundary condition as

$$\partial_t u(t,0) = -u(t,0)\rho_0 + \partial_s u(t,0)(b_0 - \gamma(0)) + \int_0^\infty \beta_0(y)u(t,y)\mathrm{d}y,$$

and we work in $\mathcal{X} = (L^1(0,\infty) \times \mathbb{R}, \|\cdot\|_{\mathcal{X}})$ with norm

$$\|(x,x_0)\|_{\mathcal{X}} = \|x\|_{L^1(0,\infty)} + c_1|x_0|.$$

Well-posedness

$$
\begin{aligned}
A_K \begin{pmatrix} u \\ u_0 \end{pmatrix} &= A \begin{pmatrix} u \\ u_0 \end{pmatrix} + K \begin{pmatrix} u \\ u_0 \end{pmatrix} \\
&= \begin{pmatrix} (du')' - (\gamma u)' - \mu u \\ (b_0 - \gamma(0))u'(0) - \rho_0 u_0 \end{pmatrix} + \begin{pmatrix} \int_0^\infty \beta(\cdot, y) u(y) \mathrm{d}y \\ \int_0^\infty \beta_0(y) u(y) \mathrm{d}y \end{pmatrix}
\end{aligned}
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with domain $D(A_K)$ given by

$$\{(u, u_0) \in \mathcal{X};\ u \in W^{2,1}_{loc}(\mathbb{R}_+),\ u(0) = u_0,\ \ (du')' - (\gamma u)' \in L^1(\mathbb{R}_+)$$
$$\text{and } \lim_{s \to \infty} d(s)u'(s) - \gamma(s)u(s) = 0\},$$

where

$$W^{2,1}_{loc}(\mathbb{R}_+) := \left\{ u \in L^1_{loc}(\mathbb{R}_+);\ u \in W^{2,1}(0, c),\ \forall c > 0 \right\}.$$

Well-posedness

Theorem

The operator A_K generates an irreducible C_0-semigroup $\{T_{A_K}(t)\}_{t\geq 0}$ in $\mathcal{X}$.

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The operator A_K generates an irreducible C_0-semigroup $\{T_{A_K}(t)\}_{t\geq 0}$ in $\mathcal{X}$.

However, in the infinite case:

- the resolvent $(\lambda I - A_K)^{-1}$ is not compact;
- we cannot use Marek's arguments, and the **spectral gap** is **not** insured.

Asynchronous exponential growth

Theorem

Suppose that there exists a measurable subset $I \subset \mathbb{R}_+$ with positive measure, such that

$$u \in L^1(\mathbb{R}_+),\ u(y) > 0 \text{ a.e.} \Longrightarrow \int_0^\infty \beta(s,y)u(y)\mathrm{d}y > 0 \text{ a.e. } s \in I.$$

If

$$\lim_{\lambda \to s(A)} r_\sigma(K(\lambda - A)^{-1}) > 1,$$

then the semigroup $\{T_{A_K}(t)\}_{t\geq 0}$ generated by A_K has the property of asynchronous exponential growth.

Asynchronous exponential growth

Sketch of proof:

1. $\omega_{\text{ess}}(\{T_{A_K}(t)\}_{t\geq 0}) = \omega_{\text{ess}}(\{T_A(t)\}_{t\geq 0}) \leq s(A)$;

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$$(s(A), \infty) \ni \lambda \mapsto r_\sigma(K(\lambda - A)^{-1})$$
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4. there exists a unique $\overline{\lambda} > s(A)$ such that
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is strictly decreasing;
4. there exists a unique $\overline{\lambda} > s(A)$ such that
$$r_\sigma(K(\overline{\lambda} - A)^{-1}) = 1 \in \sigma_p(K(\overline{\lambda} - A)^{-1});$$
5. finally, $\overline{\lambda} \in \sigma_p(A_K)$ and
$$\omega_0(\{T_{A_K}(t)\}_{t\geq 0}) = s(A_K) \geq \overline{\lambda} > s(A) \geq \omega_{\text{ess}}(\{T_{A_K}(t)\}_{t\geq 0}). \qquad \square$$

A practical criterion

Lemma

If β is bounded below by a separable kernel

$$\beta(x,y) \geq \beta_1(x)\beta_2(y),$$

where $\beta_1 \in L^1(0,\infty)$, $\beta_2 \in L^\infty(0,\infty)$ and β_1 continuous in 0, then

$$r_\sigma\left(K(\lambda - A)^{-1}\right) \geq \left\| \beta_2 \left((\lambda - A)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right)_1 \right\|_{L^1(\mathbb{R}_+)},$$

where $(\cdot)_1$ denotes the first component.

Constant case

Theorem

Suppose that

$$d \equiv 1, \quad \gamma \in \mathbb{R}, \quad \mu \in \mathbb{R}_+.$$

Let $I_1, I_2 \subset \mathbb{R}_+$ *with positive measures. Assume that*

$$\beta(x,y) \geq \beta_1(x)\beta_2(y)$$

where $\beta_1 \in L^1(0,\infty)$, $\beta_2 \in L^\infty(0,\infty)$ *are such that*

$$\beta_1(s) > 0 \text{ a.e. } s \in I_1, \quad \beta_2(s) > 0 \text{ a.e. } s \in I_2$$

with β_1 *continuous in* 0*.*

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where $\beta_1 \in L^1(0, \infty)$, $\beta_2 \in L^\infty(0, \infty)$ *are such that*

$$\beta_1(s) > 0 \text{ a.e. } s \in I_1, \quad \beta_2(s) > 0 \text{ a.e. } s \in I_2$$

with β_1 *continuous in* 0. *Then*

$$\lim_{\lambda \to s(A)} \left\| \beta_2 \left((\lambda - A)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right)_1 \right\|_{L^1} = \infty.$$

References

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Thank you for your attention !