Generators, martingale problems, and stochastic equations

Generators for Markov processes

An E-valued process is *Markov* wrt $\{\mathcal{F}_t\}$ if X is $\{\mathcal{F}_t)\}$-adapted and

$$E[f(X(t+s))|\mathcal{F}_t] = E[f(X(t+s))|X(t)] \equiv T(s)f(X(t)), \quad f \in B(E)$$

$$\begin{aligned} E[f(X(t+s+r))|\mathcal{F}_t] &= T(s+r)f(X(t)) \\ &= E[E[f(X(t+s+r))|\mathcal{F}_{t+s}]|\mathcal{F}_t] \\ &= E[T(r)f(X(t+s))|\mathcal{F}_t] \\ &= T(s)T(r)f(X(t)) \end{aligned}$$

$\{T(t), t \geq 0\}$ is a semigroup of bounded operators on $B(E)$. The *generator* for $\{T(t)\}$ satisfies

$$T(t)f = f + \int_0^t AT(s)f ds = f + \int_0^t T(s)Af ds$$

for f in a domain $\mathcal{D}(A)$.

Martingale properties

The second equality $T(t)f = f + \int_0^t T(s)Af\,ds$ can be written as

$$\begin{aligned} E[f(X(r+t))|X(r)] &= E[f(X(r+t))|\mathcal{F}_r] \\ &= f(X(r)) + E[\int_r^{r+t} Af(X(s))ds|\mathcal{F}_r] \end{aligned}$$

which in turn implies

$$E[f(X(r+t)) - f(X(r)) - \int_r^{r+t} Af(X(s))ds|\mathcal{F}_r] = 0$$

which in turn implies

$$M_f(t) = f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is a martingale, that is $E[M_f(t+r)|\mathcal{F}_r] = M_f(r)$.

This martingale property can be used to characterize the corresponding Markov process. (Stroock and Varadhan (1979))

Forward equations

Let ν_t be the distribution of $X(t)$ where X is a solution of the martingale problem for A. Then the fact that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is a martingale (and hence has expectation zero) implies

$$\nu_t f = \nu_0 f + \int_0^t \nu_s Af, \quad f \in \mathcal{D}(A),$$

$$\nu_t f = \int f d\nu_t$$

Of course, if A generates a semigroup,

$$\nu_t f = \nu_0 T(t) f$$

Examples of generators

Poisson process ($E = \{0, 1, 2 \ldots\}$, $\mathcal{D}(A) = B(E)$)

$$Af(k) = \lambda(f(k+1) - f(k))$$

Pure jump process (E arbitrary)

$$Af(x) = \lambda(x) \int_E (f(y) - f(x))\mu(x, dy)$$

Diffusion process ($E = \mathbb{R}^d$, $\mathcal{D}(A) = C_c^2(\mathbb{R}^d)$)

$$Af(x) = \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x) \frac{\partial}{\partial x_i} f(x)$$

ODE $\dot{X} = F(X)$ ($E = \mathbb{R}^d$, $\mathcal{D}(A) - C_c^1(\mathbb{R}^d)$)

$$Af(x) = F(x) \cdot \nabla f(x)$$

The martingale problem for A

X is a solution for the martingale problem for (A, ν_0), $\nu_0 \in \mathcal{P}(E)$, if $PX(0)^{-1} = \nu_0$ and there exists a filtration $\{\mathcal{F}_t\}$ such that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is an $\{\mathcal{F}_t\}$-martingale for all $f \in \mathcal{D}(A)$.

Theorem 1 *If any two solutions of the martingale problem for A satisfying $PX_1(0)^{-1} = PX_2(0)^{-1}$ also satisfy $PX_1(t)^{-1} = PX_2(t)^{-1}$ for all $t \geq 0$, then the f.d.d. of a solution X are uniquely determined by $PX(0)^{-1}$*

If X is a solution of the MGP for A and $Y_a(t) = X(a+t)$, then Y_a is a solution of the MGP for A.

Theorem 2 *If the conclusion of the above theorem holds, then any solution of the martingale problem for A is a Markov process.*

Stochastic differential equations for diffusions

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds$$

where W is a standard Brownian motion corresponds to

$$Af(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}f(x) + \sum_i b_i(x)\frac{\partial}{\partial x_i}f(x)$$

where $a(x) = \sigma(x)\sigma(x)^T$.

Stochastic equations for jump processes

$$X(t) = X(0) + \int_{[0,t]\times[0,\infty)\times[0,1]} \mathbf{1}_{[0,\lambda(X(s-))]}(u)(H(X(s-),v) - X(s-)) \times\xi(ds,du,dv)$$

where ξ is a Poisson random measure with mean measure $ds\times du\times dv$ (i.e., Lebesgue measure), corresponds to

$$Af(x) = \lambda(x)\int_E (f(y) - f(x))\mu(x,dy)$$

provided for ζ uniform $[0,1]$,

$$P\{H(x,\zeta)\in C\} = \mu(x,C)$$

Equivalence theorem: First direction

Theorem 3 *Every solution of the stochastic equation gives a solution of the martingale problem. Every solution of the martingale problem gives a solution of the forward equation.*

Proof. Itô's formula. □

Equivalence theorem: Other direction

Theorem 4 *Suppose $\mathcal{D}(A) \subset C_b$ is closed under multiplication and separates points and $(1, 0) \in A$ (plus technical conditions that are almost certainly satisfied). Then every solution of the forward equation corresponds to a solution of the martingale problem and every solution of the martingale problem corresponds to a solution of the stochastic equation.*

Proof. Existence of solutions of the martingale problem corresponding to solutions of the forward equations follows from work by Echeverría (1982); Ethier and Kurtz (1986); Bhatt and Karandikar (1993).

For diffusions, existence of solutions to stochastic equations corresponding to solutions of the martingale problem was given by Stroock and Varadhan (1979). For general Markov processes in $\mathbb{R}^d$, see Kurtz (2011). For reflecting diffusions, see Kang and Ramanan (2017). For processes whose generators can be written as an infinite sum of bounded generators, see Etheridge and Kurtz (2018). □

Constrained martingale problems

E compact (think $E = \mathbb{R}^d \cup \{\infty\}$), $E_0 \subset E$, open, A, the generator for a Markov process on E. For example,

$$Af(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x)\frac{\partial}{\partial x_i} f(x), \quad \mathcal{D}(A) = C_c^2(\mathbb{R}^d).$$

A determines the behavior of the process in E_0.

B, the generator of a Markov process (almost) which determines the behavior of the process in E_0^c and "constrains" the process to stay in $\overline{E}_0$. For example,

$$Bf(x) = \gamma(x) \cdot \nabla f(x),$$

where γ determines the direction a constraining "force" pushes when the process is on ∂E_0.

A controlled martingale problem Kurtz (1991)

Let

$$Cf(y,u,v) = vAf(y) + (1-v)Bf(y,u)$$

with controls $(u,v) \in U \times [0,1]$. We allow relaxed controls so the formulation of the martingale problem becomes

Definition 5 *(Y,μ), with $Y \in D_E[0,\infty)$, and μ a $\mathcal{P}(U \times [0,1])$-valued process, is a solution of the controlled martingale problem if there exists a filtration $\{\mathcal{F}_t\}$ such that (Y,μ) is $\{\mathcal{F}_t\}$-adapted and*

$$f(Y(t)) - f(Y(0)) - \int_0^t (V(s)Af(Y(s))ds - \int_0^t \int_{U\times[0,1]} (1-v)Bf(Y(s),u)\mu_s(du,dv))ds$$

is an $\{\mathcal{F}_t\}$-martingale, where $V(s) = \int v\mu_s(du,dv)$.

The choice of controls must be restricted so that $V(t) = 1$ if $Y(t) \in E_0$, $V(t) = 0$ if $Y(t) \in \overline{E}_0^c$, $0 \le V(t) \le 1$ if $Y(t) \in \partial E_0$.

Reflecting diffusions

Suppose

$$Af(x) = \sum_{i,j} \frac{1}{2} a_{ij}(x)\partial_i\partial_j f(x) + \sum_i b_i(x)\partial_i f(x)$$

and $Bf(x) = \kappa(x) \cdot \nabla f(x)$. Let

$$\lambda_0(t) = \int_0^t V(s)ds \qquad \lambda_1(t) = \int_0^t (1 - V(s))ds$$

Then the martingale is

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s))d\lambda_0(s) - \int_0^t \kappa(Y(s)) \cdot \nabla f(Y(s))d\lambda_1(s).$$

Of course

$$\lambda_0(t) + \lambda_1(t) = t.$$

Time change

We have martingales

$$f(Y(t))-f(Y(0))-\int_0^t Af(Y(s))d\lambda_0(s)-\int_0^t \kappa(Y(s))\cdot\nabla f(Y(s))d\lambda_1(s),$$

where $\lambda_0(t)+\lambda_1(t) = t$. If the boundary is smooth with $n(y)$ the inward normal at $y \in \partial E_0$, and $\kappa(y)\cdot n(y) > 0$, $y \in \partial E_0$, then λ_0 is strictly increasing and $\tau(t) = \inf\{s : \lambda_0(s) > t\}$ is continuous. Define $X(t) = Y(\tau(t))$ and $\lambda(t) = \lambda_1(\tau(t))$. Then

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t \kappa(X(s))\cdot\nabla f(X(s))d\lambda(s)$$

is a $\{\mathcal{F}_{\tau(t)}\}$-martingale (or perhaps a local martingale).

Controlled stochastic differential equation

The corresponding SDE should be

$$Y(t) = Y(0) + \int_0^t \sqrt{V(s)} \sum_j \sigma_{ij}(Y(s))dW_j(s) + \int_0^t V(s) \sum_i b_i(Y(s))ds + \int_0^t (1 - V(s))\kappa(Y(s))ds,$$

and by the same arguments used in Stroock and Varadhan (1979) or those in Kurtz (2011), every solution of the controlled martingale problem corresponds to a solution of the SDE.

Stochastic differential equation

Inverting λ_0 as above,

$$X(t) = X(0) + \int_0^t \sum_j \sigma_{ij}(X(s))dW_j^V(s) + \int_0^t \sum_i b_i(X(s))ds + \int_0^t \kappa(X(s))d\lambda(s),$$

where the

$$W_j^V(s) = \int_0^{\tau(t)} \sqrt{V(s)}dW_j(s)$$

are independent standard Brownian motions.

Nonlocal boundary conditions

cf. Arendt, Kunkel, and Kunze (2016) and talk by Markus Kunze on Friday

As before

$$Af(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x)\frac{\partial}{\partial x_i} f(x), \quad \mathcal{D}(A) = C_c^2(\mathbb{R}^d),$$

but now take

$$Bf(x) = \int_{E_0} (f(y) - f(x))\mu(x, dy)$$

where we assume $\mu(x, E_0) = 1$. The controlled martingale then becomes

$$f(Y(t)) - f(Y(0)) - \int_0^t V(s)Af(Y(s))ds - \int_0^t (1 - V(s))Bf(Y(s))ds.$$

or

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s))d\lambda_0(s) - \int_0^t Bf(Y(s))d\lambda_1(s).$$

Corresponding SDE

The controlled process will satisfy

$$Y(t) = Y(0) + \int_0^t \sqrt{V(s)} \sum_j \sigma_{ij}(Y(s))dW_j(s) + \int_0^t V(s) \sum_i b_i(Y(s))ds$$

$$+ \int_{[0,t]\times[0,\infty)\times[0,1]} \mathbf{1}_{[0,1]}(u)(H(Y(s-),v) - Y(s-))\xi((1-V(s))ds, du, dv)$$

As before, every solution of the controlled martingale problem corresponds to a solution of the controlled SDE, but λ_0 need not be (probably isn't) strictly increasing.

Stochastic equation for X

If, for the diffusion corresponding to A, $P\{\inf\{t : Z(t) \in \partial E_0\} = \inf\{t : Z(t) \in \overline{E}_0^c\} = 1$, then $Y(s) \in \partial E_0$ implies $V(s) = 0$ and $X(t) = Y(\tau(t))$ satisfies

$$X(t) = X(0) + \int_0^t \sum_j \sigma_{ij}(X(s))dW_j^V(s) + \int_0^t \sum_i b_i(X(s))ds$$
$$+ \int_0^t (H(X(s-), \xi_{N(s)}) - X(s-))dN(s)$$

where $N(t)$ counts the number times X hits ∂E_0 by time t, ξ_k is the v-coordinate at the kth jump time of Y (note that the ξ_k are independent, uniform $[0,1]$) and, as before,

$$W_j^V(s) = \int_0^{\tau(t)} \sqrt{V(s)}dW_j(s)$$

are independent standard Brownian motions.

Adam's question

Introduce an extra point ∂ and define

$$Bf(x) = \kappa(x) \cdot \nabla f(x) + c(x)(f(\partial) - f(x)).$$

Then the process satisfies

$$X(t) = X(0) + \int_0^t \sum_j \sigma_{ij}(X(s))dW_j^V(s) + \int_0^t \sum_i b_i(X(s))ds + \int_0^t \kappa(X(s))d\lambda(s),$$

until a killing time γ satisfying

$$\int_0^\gamma c(X(s))d\lambda(s) = \Delta,$$

where Δ is an independent unit exponential random variable.

The corresponding martingale problem

X is a solution of the martingale problem if there exists a filtration $\{\mathcal{F}_t\}$ and a nondecreasing process λ such that λ increases only when $X(s) \in \partial E_0$ and for each $f \in C_c^2(\mathbb{R}^d \cup \{\partial\})$, taking $Af(\partial) = 0$,

$$\begin{aligned} f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds \\ - \int_0^t (\kappa(X(s)) \cdot \nabla f(X(s)) + c(X(s))(f(\partial) - f(X(s)))d\lambda(s) \end{aligned}$$

is an $\{\mathcal{F}_t\}$-martingale.

Viscosity solutions for the H-Y range condition

Costantini and Kurtz (2015)

Let Y be a solution of the controlled martingale problem

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s))d\lambda_0(s) - \int_0^t Bf(Y(s))d\lambda_1(s),$$

and assuming $\lambda_0(s) \to \infty$, let $\tau(t) = \inf\{s : \lambda_0(s) > t\}$, and define

$$X(t) = \lim_{s \to t+} Y(\tau(s)).$$

Then $\int_0^\infty e^{-\lambda_0(s)} f(Y(s))d\lambda_0(s) = \int_0^\infty e^{-t} f(X(t))dt$ and for $Y(0) = x$, we should have

$$E[\int_0^\infty e^{-\lambda_0(s)} f(Y(s))d\lambda_0(s)] = E[\int_0^\infty e^{-t} f(X(t))dt] = (I - \widehat{A})^{-1} f(x)$$

where $\widehat{A}$ is the generator for X,

$$\widehat{A} \supset \{(f, Af) : Bf = 0\}$$

Sub and super solutions

Let Π_x be the collection of distributions of solutions of the controlled martingale problem (Y, λ_0). Then assuming that for all $x \in \overline{E}_0$, Π_x is nonempty and compact and ...

$$u_h^+(x) = \sup_{P \in \Pi_x} E[\int_0^\infty e^{-\lambda_0(s)} h(Y(s)) d\lambda_0(s)]$$

is a subsolution of $(I - \widehat{A})u = h$ in the sense that u^+ is upper semi-continuous and if $f \in \mathcal{D}$ and $x_0 \in \overline{E}_0$ satisfy

$$\sup_x (u^+ - f)(x) = (u^+ - f)(x_0), \tag{1}$$

then

$$\begin{aligned} \lambda u^+(x_0) - Af(x_0) \leq h(x_0), & \qquad \text{if } x_0 \in E_0, \\ (\lambda u^+(x_0) - Af(x_0) - h(x_0)) \wedge (-Bf(x_0)) \leq 0, & \qquad \text{if } x_0 \in \partial E_0. \end{aligned}$$

Notions of convergence

$(Y^n, \lambda_0^n) \to (Y, \lambda_0)$ in distribution on $(D_E[0,\infty) \times C_{[0,\infty)}[0,\infty)$ taking the Skorohod (J_1) topology on $D_E[0,\infty)$ and the compact uniform topology on $C_{[0,\infty)}[0,\infty)$.

Lemma 6 *If* $(Y^n, \lambda_0^n) \to (Y, \lambda_0)$ *as above then* $X^n \to X$ *taking the Jakubowski (1997) topology on* $D_E[0,\infty)$*.*

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Abstract

Generators, martingale problems, and stochastic equations

A natural way of specifying a Markov process is by defining its generator. Classically, one then shows that the generator, or some natural extension, is the generator of a positive, contraction semigroup which determines the transition function of the Markov process which must satisfy the Kolmogorov *forward* and *backward* equations.

Standard semigroup identities and the relationship between the process and the semigroup also imply that the process has certain martingale properties which are the basis for the classical identity known as Dynkin's identity. In work on diffusions, Stroock and Varadhan, exploiting these properties, formulated a *martingale problem* as an approach to uniquely determining the process corresponding to the generator.

For many processes, in particular diffusions, the process can also be determined as a solution of a stochastic equation. Very generally, the forward equation, the martingale problem, and, if one exists, the corresponding stochastic equation, are equivalent in the sense that a solution of one corresponds to solutions of the others. In particular, uniqueness of one implies uniqueness of the others.

The formulation of the three problems becomes more complicated in the case of constrained Markov processes (for example, reflecting diffusions). Associating a constrained martingale problem with a certain controlled martingale problem in a sense reduces the problem of equivalence of the three approaches to specifying the process to the unconstrained case. The forward equations, martingale problems, and (in examples) stochastic equations will be formulated and proof of their equivalence outlined.