Implicit Fokker-Planck Equations: Non-commutative Convolution of Probability Distributions

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The Problem: Two state space transition functions

[sWSLS18] Two-space approach to absorbing boundary of a Markov process leads to two distinct types of discrete state spaces:

- $\mathbb{N}_X := \{1, 2, ..., m\}$, the set of m "life" states, and
  $\mathbb{N}_{\bar{Y}} := \{\bar{1}, \bar{2}, ..., \bar{n}\}$, the set of n "death" states.

Continuous analogues of $\mathbb{N}_X$ and $\mathbb{N}_{\bar{Y}}$: [distinct copies of $\mathbb{R}$]

- Continuums of life and death (cemetary ∂) states $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$.

Thought Experiment: Two Pipes separated permeable membrane.

Two types intertwining by possibility of transitioning

- (a) from a life state to another life state within $\mathbb{R}_X$
- (b) from a life state in $\mathbb{R}_X$ to a death state in $\mathbb{R}_{\bar{Y}}$.

The Setting/ Pitfall of Feller Convolution

Two types of Stochastic kernels:

- (a) one-space stochastic kernel$Q(x, B)$
- (b) **Uni-directional** *two-space* stochastic kernel $R(x, \bar{B})$

Homogeneous Markov processes $(\mathbf{X}, \mathbf{Y})$ intertwined BW-ECK [1] :

$$Q_{t+s}(x, B) = \int_{y \in \mathbb{R}} Q_t(x, \{dy\}) Q_s(y, B); \qquad (1a)$$

$$R_{t+s}(x, \bar{B}) = \int_{y \in \mathbb{R}} Q_t(x, \{dy\}) R_s(y, \bar{B}). \qquad (1b)$$

- Function $R_s(y, \bar{B})$ runs intermediary transition points y to set $\bar{B}$.
- Integration with respect to the same life measure $Q_t(x, \{dy\})$.

[1] Operator Represent is reverse Empathy: $S(t+s) = E(t)S(s)$ [sWSLS18].

Noncommutative Convolution Needed!

The two-space backward transition equation (1b) can be expressed in terms of the pair of *distribution* transition functions $(\mathbf{Q}, \mathbf{R})$ [2]

$$R_{t+s}\{\bar{B}\} := R_{t+s}(0, \bar{B}) = \int_{y \in \mathbb{R}} Q_t\{dy\} R_s(0, \bar{B} - y)$$

If no distinction between $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$:

$$R_{t+s}\{d\bar{y}\} = Q_t\{dy\} \star R_s\{d\bar{y}\} = R_s\{d\bar{y}\} \star Q_t\{dy\} \qquad (2)$$

Last equality (by **commutativity** of Feller convolution) is nonsense.

- Language of distributions inadequate: replace distributions with admissible homomorphisms.

[2] $(\mathbf{Q}, \mathbf{R}) := (Q_t\{dy\}, R_t\{d\bar{y}\})_{t>0} = (Q_t(0, \{dy\}, R_t(0, \{d\bar{y}\})_{t>0}$.

Admissible Homomorphisms/Generalized Operators

Feller represent $Q \mapsto (Q' : \Phi \to \mathbb{C}) \mapsto (\mathcal{Q} := \Gamma(Q') : \Phi \to \Phi)$.

$$\langle Q', f \rangle = Q'(f) := \int_{\mathbb{R}} Q\{dy\} f(y) \text{ for all } f \in \Phi. \tag{3}$$

$$\mathcal{Q} f(x) = [Q \circledast f](x) := \langle Q', f_{-x} \rangle = \int_{\mathbb{R}} Q\{dy\} f(x+y) \tag{4}$$

Riesz representation Q' is a Φ-*admissible homomorphism* (Set $\mathscr{A}_\Phi$)

$$Q \circledast f \in \Phi \text{ for all } f \in \Phi, \tag{5}$$

Test space: $\Phi_U = \mathrm{BUC}(\mathbb{R}, \mathbb{C})$, $\Phi_0 := C_0(\mathbb{R}, \mathbb{C})$, $\Phi_\infty := C[\mathbb{R}, \mathbb{C}]$
Associative Product $*$ on $\mathscr{A}_\Phi$ (convolution algebra) is defined by

$$\langle Q_1' * Q_2', f \rangle = \langle Q_1', \mathcal{Q}_2 f \rangle \text{ for all } f \in \Phi. \tag{6}$$

Feller convolution of distributions $[Q \star R](x) = \int_{\mathbb{R}} Q\{dy\} R(x+y)$.

$$[Q \star R]' = Q' * R'; \Gamma(Q' * R') = \mathcal{Q} \circ \mathcal{R}. \tag{7}$$

Convolution Product $*$ Replaces Feller convolution

Theorem

Each admissible homomorphism represents a unique distribution, i.e., the mapping $Q \mapsto Q'$ is injective. Convolution of distributions lifts as the product of admissible homomorphisms (7).

[Feller convolution $\star$] Distribution transition function $\mathbf{Q}$ intertwined by C-K equation (1a) is a Feller convolution semigroup:

$$Q_{t+s}\{dy\} = Q_t\{dy\} \star Q_s\{dy\} \ \textit{for all} \ s, t > 0; \qquad (8)$$

$$\mathcal{Q}_{t+s} = \mathcal{Q}_t \circ \mathcal{Q}_s \qquad \textit{for all} \ s, t > 0. \qquad (9)$$

[Theorem 1] Replace $\mathbf{Q}$ by time continuum, $\mathfrak{q}' := \{Q'_t\}_{t>0}$, of admissible homomorphisms on Φ_U ("*admissible transition function*"). Then Feller convolution semigroup is a star-semigroup:

Star Semigroup Replaces Feller Convolution Semigroup

Admissible transition function $\mathfrak{q}'$ is a star-semigroup:

$$Q'_{t+s} = Q'_t * Q'_s \ \text{for all } s, t > 0; \tag{10}$$

$$\mathcal{Q}_{t+s} = \mathcal{Q}_t \circ \mathcal{Q}_s \ \text{for all } s, t > 0. \tag{11}$$

For Convolution product $*$ to be required non-commutative extension of the Feller convolution, use versatility of framework of admissible homomorphisms is freedom to change test spaces.

- Hack 1: Product Test space with Diagonal Group
- Hack 2: Dual FWECK: Representation of uni-directional dual FWECK (cf (1b)) as a star empathy.
- Hack 3: Star Empathy machinery generates implicit convolution Fokker-Planck equation (IFP).

Forward Extended Chapman-Kolmogorov Equation

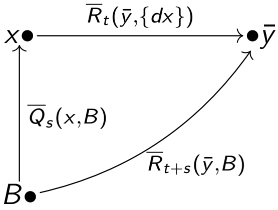
Assume kernels $Q_t(x, B)$ and $R_t(x, \bar{B})$ (of BWECK (1)) have probability transition *density* functions $q_t(x, y)$ and $r_t(x, \bar{y})$. Construct *conjugate kernels* $\bar{Q}_t(y, B)$ and $\bar{R}_t(\bar{y}, B)$:

$$\bar{Q}_t(y, B) := \int_{x \in B} q_t(x, y) dx, \bar{R}_t(\bar{y}, B) := \int_{x \in B} r_t(x, \bar{y}) dx, . \quad (12)$$

Then conjugate transition functions $\bar{Q}_t(y, B)$ and $\bar{R}_t(\bar{y}, B)$ satisfy the *forward extended Chapman-Kolmogorov equation*

$$\bar{Q}_{t+s}(y, B) = \int_{x \in \mathbb{R}} \bar{Q}_t(y, \{dx\}) \bar{Q}_s(x, B) \text{ for all } s, t > 0; \quad (13a)$$

$$\bar{R}_{t+s}(\bar{y}, B) = \int_{x \in \mathbb{R}} \bar{R}_t(\bar{y}, \{dx\}) \bar{Q}_s(x, B) \text{ for all } s, t > 0. \quad (13b)$$



Distribution transition functions **Q** and **R** defined on distinct spaces, $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$. Conjugation operation produces a corresponding pair of distribution transition functions on $\mathbb{R}_X$.

$$(\bar{\mathbf{Q}}, \bar{\mathbf{R}}) := (\bar{Q}_t\{dy\}, \bar{R}_t\{dy\})_{t>0} = (\bar{Q}_t(0, \{dy\}), \bar{R}_t(\bar{0}, \{dy\}))_{t>0}$$

$(\bar{\mathbf{Q}}, \bar{\mathbf{R}})$ expresses two-space forward transition equation (13b):

$$\bar{R}_{t+s}\{B\} := \bar{R}_{t+s}(\bar{0}, B) = \int_{x\in\mathbb{R}} \bar{R}_t\{dy\}\bar{Q}_s(0, B-x). \qquad (14)$$

Star empathy in a product test space

Let $\Phi_X := BUC(\mathbb{R}_X, \mathbb{C})$ and $\Phi_{\bar{Y}} := BUC(\mathbb{R}_{\bar{Y}}, \mathbb{C})$.

- Diagonal additive group $G := \{(\sigma, \sigma) | \sigma \in \mathbb{R}\}$ for one parameter shifts. For each $(f, \bar{g}) \in \Phi_X \times \Phi_{\bar{Y}}$, define corresponding test function $\varphi : G \to \mathbb{C}^2 := \mathbb{C} \times \mathbb{C}$ by $\varphi(\sigma, \sigma) = (f(\sigma), \bar{g}(\bar{\sigma}))$.

Product test space Φ_P is set of all such functions φ.**Admissible linear functionals are** $\mathbb{C}^2 := \mathbb{C}'life' \times \mathbb{C}'death'$**-valued.**
Replace $Q_t\{dy\}$ on $\mathbb{R}_X$ and $R_t\{d\bar{y}\}$ on $\mathbb{R}_{\bar{Y}}$ by admissible homomorphisms $Q_t'(X)$ on Φ_X and $R_t'(\bar{Y})$ on $\Phi_{\bar{Y}}$. Lift single space homomorphisms as product space homomorphisms by:

- liftings $\ell_1 : \mathbb{C} \to \mathbb{C}^2 : z \mapsto (z, 0)$ and $\ell_2 : \mathbb{C} \to \mathbb{C}^2 : z \mapsto (0, z)$;
- liftings $\ell_X : \Phi_X \to \Phi_P : f \mapsto (f, 0_{\bar{Y}})$ and $\ell_{\bar{Y}} : \Phi_{\bar{Y}} \to \Phi_P : f \mapsto (0_X, \bar{f})$.
- the projection $\pi_X : \Phi_P \to \Phi_X : \varphi = (f, \bar{g}) \mapsto f$.

Lift $\bar{Q}'_t(X)$ and $\bar{R}'_t(X)$ as the Φ_P-admissible homomorphisms

$$Q'_P(t) := \ell_1 \circ \bar{Q}'_t(X) \circ \pi_X, \quad R'_P(t) := \ell_2 \circ \bar{R}'_t(X) \circ \pi_X$$

Injection into "life" ["death"] part of $\mathbb{C}^2$ gives life ["death"] dualism function. Then, corresponding pair of conjugate Φ_P-admissible transition functions $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P) := (\bar{Q}'_P(t), \bar{R}'_P(t))_{t>0}$.

Theorem

Let $(\mathbf{X}, \mathbf{Y})$ be a pair of homogeneous Markov processes intertwined by BWECK (1). Then, in terms of the product $*$, pair of conjugate Φ_P-admissible transition functions $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ is star-empathy:

$$\bar{Q}'_P(t+s) = \bar{Q}'_P(t) * \bar{Q}'_P(s); \tag{15a}$$
$$\bar{R}'_P(t+s) = \bar{R}'_P(t) * \bar{Q}'_P(s). \tag{15b}$$

Moreover, $\bar{Q}'_P(s) * \bar{R}'_P(t)$ is the zero homomorphism on Φ_P.

Convolution semigroup with a single test space

Framework of Admissible homomorphisms has fully developed Laplace transform theory: Laplace transform approach to generators.

- Laplace transform theory requires the strong continuity of $\bar{\mathfrak{q}}'_P$ and $\bar{\mathfrak{r}}'_P$.

Consider single homogeneous Markov process $\mathbf{X}$ with admissible transition function $\mathfrak{q}' = \{Q'_t\}_{t>0}$.

- Extra initial condition $\lim_{t \to 0^+} Q_t\{dy\} = \delta_0$ to $\mathbf{Q}$

Then $\mathfrak{q}'$ is a strongly continuous star-semigroup and the dualism transition function $\mathfrak{Q} := \{\mathcal{Q}_t\}_{t>0}$, where $\mathcal{Q}_t = \Gamma(Q'_t)$, is an operator C_0-semigroup.

- The distribution transition function $\mathbf{Q}$ is defective (BWECK).

Defective convolution semigroup

Proposition

Let $\mathbf{Q}$ be a convolution semigroup and defective. Then there exist a unique $c > 0$ and a unique distribution transition function $\mathbf{P} = \{P_t\{dy\}\}_{t>0}$ that is a convolution semigroup with proper distributions such that $Q_t\{dy\} = e^{-ct}P_t\{dy\}$. The Feller generator of $\mathbf{Q}$ is $A - cI$, where A is the Feller generator of $\mathbf{P}$.

$\mathbf{P}$ is the transition distribution function associated with the standard Brownian motion. Then we call $\mathbf{X}$ a *defective Brownian motion*. In this case the Feller generator A of $\mathbf{Q}$ is given by

$$\bar{A}f = \frac{1}{2}f'' - cf \text{ for all } f \in C^\infty[\mathbb{R}, \mathbb{C}]. \qquad (16)$$

Preservation by Conjugation

FWECK stated purely in terms of conjugate transition kernels:

- If $\mathbf{Q}$ is a convolution semigroup, then so is $\bar{\mathbf{Q}}$: $\bar{\mathfrak{q}}'$ is a strongly continuous star-semigroup and the $\mathrm{dualism}$ transition function $\bar{\mathfrak{Q}} := \{\bar{\mathcal{Q}}_t\}_{t>0}$, where $\bar{\mathcal{Q}}_t = \Gamma(\bar{Q}'_t)$, is an operator C_0-semigroup.
- If $\mathbf{Q}$ is defective, then so too is $\bar{\mathbf{Q}}$.
- Strong continuity of $\bar{\mathfrak{r}}'_P$ follows from strong continuity of $\bar{\mathfrak{q}}'_P$ $[(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ is a star-empathy].
- The Feller generator $\bar{A}$ of $\bar{\mathbf{Q}}$ is given by

$$\bar{A}f = \frac{1}{2}f'' - cf \text{ for all } f \in C^\infty[\mathbb{R}, \mathbb{C}]. \tag{17}$$

Easily extend to $\bar{\mathfrak{q}}'_P$

Conjugate convolution semigroup with a product test space

State spaces $\mathbb{R}_X$ and $\mathbb{R}_{\bar{Y}}$ are distinct copies of $\mathbb{R}$: consider two distinct test spaces

$$\Phi_\infty(X) := C[\mathbb{R}_X, \mathbb{C}] \subset \Phi_X, \quad \Phi_\infty(\bar{Y}) := C[\mathbb{R}_{\bar{Y}}, \mathbb{C}] \subset \Phi_{\bar{Y}}.$$

BWECK (1b) requires the product test space Φ_P associated with $\Phi_\infty(X) \times \Phi_\infty(\bar{Y})$. Here $\theta_0'(X) := \ell_1 \circ \theta_0' \circ \pi_X$, i.e.,

$$\langle \theta_0'(X), \varphi \rangle = (\langle \theta_0', f \rangle, 0) = (f(0), 0) \text{ for all } \varphi = (f, \bar{g}) \in \Phi_P.$$

Proposition

If $\mathbf{Q}$ *is a convolution semigroup, then* $\bar{\mathfrak{q}}_P'$ *is strongly continuous:*

$$\langle \bar{Q}_P'(t), \varphi \rangle \to \langle \theta_0'(X), \varphi \rangle \text{ as } t \to 0^+ \text{ for all } \varphi \in \Phi_P. \qquad (18)$$

Setting stage for Laplace Transform Approach

[$\bar{Q}_t\{dy\}, \bar{R}_t\{dy\}$ *are probability measures on* $\mathbb{R}_X$] $\bar{\mathfrak{q}}'_P$ is Laplace-closed w.r.t itself and that $\bar{\mathfrak{r}}'_P$ is Laplace-closed w.r.t $\bar{\mathfrak{q}}'_P$. [$(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ *is a star-empathy*]Strong continuity of $\bar{\mathfrak{q}}'_P$ ensures strong continuity of $\bar{\mathfrak{r}}'_P$,

Theorem

The conjugate extended Riesz representation on Φ_P *of* $(\mathbf{Q}, \mathbf{R})$ *satisfies the star pseudo-resolvent equations*

$$\bar{\mathfrak{q}}'_P(\lambda) - \bar{\mathfrak{q}}'_P(\mu) = (\mu - \lambda)\bar{\mathfrak{q}}'_P(\lambda) * \bar{\mathfrak{q}}'_P(\mu); \tag{19a}$$

$$\bar{\mathfrak{r}}'_P(\lambda) - \bar{\mathfrak{r}}'_P(\mu) = (\mu - \lambda)\bar{\mathfrak{r}}'_P(\lambda) * \bar{\mathfrak{q}}'_P(\mu) \tag{19b}$$

$$\bar{\mathfrak{q}}'_P(\lambda) * \bar{Q}'_P(t) = \bar{Q}'_P(t) * \bar{\mathfrak{q}}'_P(\lambda); \tag{19c}$$

$$\bar{\mathfrak{r}}'_P(\lambda) * \bar{Q}'_P(t) = \bar{R}'_P(t) * \bar{\mathfrak{q}}'_P(\lambda). \tag{19d}$$

Defective Brownian motion: λ-potential operator

For $\lambda > 0$, let

$$\bar{\mathcal{Q}}_P(\lambda) := \Gamma(\bar{\mathfrak{q}}'_P(\lambda)), \quad \bar{\mathcal{R}}_P(\lambda) := \Gamma(\bar{\mathfrak{r}}'_P(\lambda))$$

be the dualisms of the Laplace transforms. Then $\bar{\mathcal{Q}}_P(\lambda)$ is the λ-potential operator or resolvent operator.

Let $(\mathbf{Q}, \mathbf{R})$ be as in Proposition 2 with $\mathbf{Q}$ (defective Brownian).

Then Feller generator of $\bar{\mathfrak{Q}}'_P$ on $\Delta_P := C^\infty[\mathbb{R}_X, \mathbb{C}] \times \Phi_\infty(\bar{Y})$:

$$\bar{A}_P\varphi = \left(\tfrac{1}{2}f'' - cf, 0_{\bar{Y}}\right) \text{ for all } \varphi := \left(f, \bar{g}\right) \in \Delta_P. \qquad (20)$$

Moreover, for all $\varphi := \left(f, \bar{g}\right) \in \Delta_P$,

$$[\bar{\mathcal{Q}}_P(\lambda)\varphi](x,x) = \int_0^\infty e^{-(\lambda+c)t}[p_t * f](x)dt \text{ for all } x \in \mathbb{R}, \quad (21)$$

where $p_t(y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right)$ is the probability density function of the standard Brownian motion.

Example: Defective Brownian motion with a product test space

Moreover, the Φ_P-$\mathrm{dualism}$ transition function $\bar{\mathfrak{Q}}_P := \{\bar{\mathcal{Q}}_P(t)\}_{t>0}$, where $\bar{\mathcal{Q}}_P(t) = \Gamma(\bar{Q}'_P(t))$, is an operator C_0-semigroup on Φ_P. Let $(\mathbf{Q}, \mathbf{R})$ be as above with $\mathbf{Q}$ (defective Brownian). Then the Feller generator of $\bar{\mathfrak{Q}}'_P$ is defined on $\Delta_P := C^\infty[\mathbb{R}_X, \mathbb{C}] \times \Phi_\infty(\bar{Y})$

$$\bar{A}_P\varphi = \left(\tfrac{1}{2}f'' - cf, 0_{\bar{Y}}\right) \ for\ all\ \varphi := (f, \bar{g}) \in \Delta_P. \tag{22}$$

Since $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$ is a star-empathy, the strong continuity of $\bar{\mathfrak{q}}'_P$ ensures the strong continuity of $\bar{\mathfrak{r}}'_P$, i.e., for each $\varphi \in \Phi_P$, the mappings $t \mapsto \langle \bar{Q}'_P(t), \varphi\rangle$ and $t \mapsto \langle \bar{R}'_P(t), \varphi\rangle$ from $(0, \infty)$ to $\mathbb{C}^2$ are continuous. By the continuity of the dualism mapping Γ, this also implies the strong continuity of the corresponding dualism families, $\bar{\mathfrak{Q}}_P$ and $\bar{\mathfrak{R}}_P := \{\bar{\mathcal{R}}_P(t)\}_{t>0}$, where $\bar{\mathcal{R}}_P(t) = \Gamma(\bar{R}'_P(t))$.

Convolution Implicit Fokker-Planck equations

[sWSLS18], Laplace transform derived implicit Fokker-Planck equations (IFP) for BWECK-intertwined counting processes. [Unknowns = operator representations of the transition functions.]
Homogeneity: formulation of IFP in framework of admissible homomorphism: IFP directly in terms of the distributions.

- *Intertwined Brownian motion* $(\mathbf{X}, \mathbf{Y})$: $(\mathbf{X}, \mathbf{Y})$, $(\mathbf{Q}, \mathbf{R})$ and $(\bar{\mathfrak{q}}'_P, \bar{\mathfrak{r}}'_P)$; $\mathbf{X}$ is a defective Brownian.

Each fixed $\varphi = \left(f, \bar{g}\right) \in \Delta_P$, define u_P, v_P from $(0, \infty) \times \mathbb{R}$ to $\mathbb{C}^2$;

$$v_P(t,x) = [\bar{\mathcal{Q}}_P(t)\varphi](x,x), \quad u_P(t,x) = [\bar{\mathcal{R}}_P(t)\varphi](x,x). \qquad (23)$$

Thus $v_P(t,x) = (v(t,x), 0)$, where $v(t,x) = [\bar{\mathcal{Q}}_t(X)f](x)$.
[Eq. (22)] v_P satisfies convolution FP equation $\frac{\partial v_P}{\partial t} = \bar{A}_P v_P$:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}_X} \bar{Q}_t\{dy\} f(x+y) = \frac{1}{2}\int_{\mathbb{R}_X} \bar{Q}_t\{dy\}\left(\frac{\partial^2}{\partial x^2} - c\right) f(x+y). \quad (24)$$

Invertibility Assumption

Laplace transforms shows $u_P(t,x)$ satisfies a IFP as an implicit evolution equation in terms of admissible homomorphisms.
We do not derive the (convolution) star implicit evolution equation directly from the intertwined pseudo-resolvent $(\bar{\mathfrak{q}}'_P(\lambda), \bar{\mathfrak{r}}'_P(\lambda))_{\lambda>0}$.
Instead, by dualism mapping Γ to equations (25)–(25d), we obtain analogous equations for the operator-valued dualisms $(\bar{\mathcal{Q}}_P(\lambda), \bar{\mathcal{R}}_P(\lambda))_{\lambda>0}$:

$$\bar{\mathcal{Q}}'_P(\lambda) - \bar{\mathcal{Q}}'_P(\mu) = (\mu - \lambda)\bar{\mathcal{Q}}'_P(\lambda) \circ \bar{\mathcal{Q}}'_P(\mu); \tag{25a}$$

$$\bar{\mathcal{R}}'_P(\lambda) - \bar{\mathcal{R}}'_P(\mu) = (\mu - \lambda)\bar{\mathcal{R}}'_P(\lambda) \circ \bar{\mathcal{Q}}'_P(\mu) \tag{25b}$$

$$\bar{\mathcal{Q}}_P(\lambda) \circ \bar{\mathcal{Q}}_P(t) = \bar{\mathcal{Q}}_P(t) \circ \bar{\mathcal{Q}}'_P(\lambda); \tag{25c}$$

$$\bar{\mathcal{R}}'_P(\lambda) \circ \bar{\mathcal{Q}}_P(t) = \bar{\mathcal{R}}_P(t) \circ \bar{\mathcal{Q}}'_P(\lambda). \tag{25d}$$

Assume that

$$\bar{\mathcal{R}}_P(\xi) \text{ is invertible for some } \xi > 0. \tag{26}$$

Computational Power of $L_1(\mathscr{A}_\Phi)$

Then $\bar{\mathcal{R}}_P(\lambda)$ is invertible for all $\lambda > 0$. For $\lambda > 0$, let

$$\Delta_X := \bar{\mathcal{Q}}_P(\lambda)[\Phi_P], \quad \Delta_{\bar{Y}} := \bar{\mathcal{R}}_P(\lambda)[\Phi_P].$$

Furthermore, the operators A and B from $\Delta_{\bar{Y}}$ to Φ_P defined by

$$B = \bar{\mathcal{Q}}_P(\lambda)[\bar{\mathcal{R}}_P(\lambda)]^{-1}, \quad A = \lambda B - [\bar{\mathcal{R}}_P(\lambda)]^{-1},$$

where $A = \bar{A}'_P B$, where $\bar{A}'_P$ is the Feller generator of $\bar{\mathfrak{Q}}_P = \{\bar{\mathcal{Q}}_P(t)\}_{t>0}$, and that for each $\varphi = \left(f, 0_{\bar{Y}}\right) \in \Delta_X \cap \Delta_P$,

$$\frac{\partial}{\partial t}(Bu_P) = Au_P; \tag{27a}$$

$$\lim_{t\to 0^+} Bu_P(t,x) = \varphi(x,x), \ x \in \mathbb{R}. \tag{27b}$$

Backtrack to Convolution Equations

The implicit evolution equation (27) can be expressed in terms of the admissible homomorphisms $A' = \theta_0'(X) \circ A$ and $B' = \theta_0'(X) \circ B$:

$$\frac{d}{dt}\langle B' * \bar{R}_P'(t), \varphi\rangle = \langle A' * \bar{R}_P'(t), \varphi\rangle \quad \text{for a.e. } t > 0; \qquad (28a)$$

$$\lim_{t \to 0^+}\langle B' * \bar{R}_P'(t), \varphi\rangle = \langle \theta_0'(X), \varphi\rangle. \qquad (28b)$$

In terms of the original homomorphisms $\bar{R}_t'(X)$, the second component of eq. (28a) is simply $0_{\bar{Y}} = 0_{\bar{Y}}$ and the first component is

$$\frac{\partial}{\partial t}\langle B' * \bar{R}_t'(X), f_{-x}\rangle = \langle B' * \bar{R}_t'(X), \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - c\right) f_{-x}\rangle \quad \text{for a.e. } t > 0. \qquad (29)$$

Note that $\bar{Q}_t'(X) = B' * \bar{R}_t'(X)$ on $\Delta_X \cap \Delta_P$.

References

W.-S. Lee and N. Sauer, *Intertwined markov processes: the extended chapman-kolmogorov equation*, Proc. R. Soc. Edinb. A **148A** (2018), 123–131.