Emergence of non-ergodic, conservative dynamics

Pierre Berger, with two works in progress:
one with Jairo Bochi and one with Dimitry Turaev

Recently [Be16], among some open subsets of differentiable dynamical systems of compact manifold M, the coexistence of infinitely many attractors has been shown to be typical in the sense of Kolmogorov.

To describe the complexity of such dynamics, the following notion has been introduced in [Be17]:

Definition 1. *Given a standard distance on the probability measures of a space (such as the Wasserstein metric), The Emergence $\mathcal{E}(\epsilon)$ at scale $\epsilon > 0$ of a system is the minimal number N of probability measures $(\mu_i)_{1\le i\le N}$ necessarily so that the Birkhoff average $S_k(x) := \frac{1}{k}\sum_{i=1}^{k}\delta_{f^i(x)}$ satisfies:*

$$\limsup_{k\to\infty}\int_M d_{W_1}(S_k(x), \{\mu_i : 1 \le i \le N\})dLeb < \epsilon\ ,$$

where d_{W_1} is the 1-Wasserstein metric on the space of probability measures of M.

In [Be17], it has been conjectured that in some open sets of dynamical systems, typical dynamics display a super polynomial emergence:

$$\limsup_{\epsilon\to 0}\frac{\log\mathcal{E}(\epsilon)}{-\log\epsilon} = \infty\ .$$

In a work in progress with Jairo Bochi, we showed that in the open set of conservative, surface mapping diffeomorphisms displaying an elliptic point, a C^∞-generic diffeomorphism displays a maximal emergence (which is super-polynomial):

$$\limsup_{\epsilon\to 0}\frac{\log\mathcal{E}(\epsilon)}{\epsilon^2} > 0\ .$$

In a work in progress with Jairo Bochi, we showed that in the open set of hamiltonian diffeomormhisms with a totally elliptic point, a typical diffeomorphisms in the sense of Kolmogorov (i.e. Lebesgue a.e. map in a generic family) displays a maximal emergence:

$$\limsup_{\epsilon \to 0} \frac{\log \mathcal{E}(\epsilon)}{\epsilon^{2n}} > 0 \ .$$

This proves this conjecture in the Hamitonian (and surface, conservatif) context.

Furthermore, we constructed with Jairo Bochi an analogous of the variational principle of the entropy for the concept of emergence.

References

[Be16] Berger, P. – P. Berger, Generic family with robustly infinitely many sinks, *Inventiones Matematicae*, (2016) 205 : 121. 41p.

[Be17] Berger, P. – Emergence and non-typicality of the finiteness of the attractors in many topologies *Proc. Steklov Inst. Math.* 297 (2017), no. 1, 1–27.