

QUADRATIC HARNESES AS A LINK BETWEEN ASKEY-WILSON POLYNOMIALS AND ASEP_s

Jacek Wesołowski

18 WORKSHOP, BĘDLEWO 2018

Abstract

Quadratic harnesses are classical stochastic processes (X_t) determined by linear conditional expectations and quadratic conditional variances where conditioning is with respect to the past-future filtration of the process. Typically they are non-stationary Markov processes uniquely determined by five numerical constants, say $\theta, \eta, \tau, \sigma$ and γ . To a large extent their properties are determined by respective families of orthogonal martingale polynomials $(p_n(x, t))$ whose Jacobi matrices \mathbf{J}_t are affine in the time variable t , i.e. $\mathbf{J}_t = t\mathbf{X} + \mathbf{Y}$ and infinite tridiagonal matrices \mathbf{X} and \mathbf{Y} satisfy the following γ -commutation equation : $\mathbf{X}\mathbf{Y} - q\mathbf{Y}\mathbf{X} = \mathbf{I} + \theta\mathbf{X} + \eta\mathbf{Y} + \tau\mathbf{X}^2 + \sigma\mathbf{Y}^2$. The family of quadratic harnesses is rather wide and includes e.g. : Wiener process, Poisson process, Levy-Meixner processes, quantum Bessel process, free Brownian motion, q -Gaussian process, Dirichlet process, bi-Poisson processes and others. Basic facts on quadratic harnesses will be presented in the lecture. In particular we will explain a connection between quadratic harnesses and Askey-Wilson processes which have been defined in 2010 as Markov processes with marginal distributions and transition probabilities being probability measures which orthogonalize Askey-Wilson system of polynomials. This connection in particular introduces a new parametrization of Askey-Wilson polynomials - the new parameters are the five numerical constants $\theta, \eta, \tau, \sigma$ and γ identified above.

ASEP is an acronym for asymmetric simple exclusion process and describes evolution of particles which are forced to move at random times one step to the left or right in sites $1, \dots, N$ but only when the neighbouring site is not occupied ; additionally, particles are injected randomly from the left and right or can leave the system through border sites 1 or N . Then site occupations is a Markov process with the state space $\{0, 1\}^N$. We will be interested in stationary (steady state) distribution of this process, i.e. the law of the site occupation (τ_1, \dots, τ_N) after a long run. It is well known that this distribution (actually, its generating function) can be determined through so called matrix ansatz formulated by Derrida, Evans, Hakim and Pasquier in 1993. This result is based on non-commutative calculations involving two infinite matrices \mathbf{D} and \mathbf{E} satisfying identity $\mathbf{D}\mathbf{E} - q\mathbf{E}\mathbf{D} = \mathbf{D} + \mathbf{E}$, where $q \in [0, 1)$ is the intensity of jumps to the left, while 1 is the intensity of jumps to the right. This identity for \mathbf{D} and \mathbf{E} and the γ -commutation equation for \mathbf{X} and \mathbf{Y} (elements of the Jacobi matrix \mathbf{J}_t) form a theoretical base for the connection between ASEP_s and quadratic harnesses (more precisely, Askey-Wilson processes) we are going to describe in the lecture. Actually, we will explain how the generating function of site occupations (τ_1, \dots, τ_N) can be represented in terms of moments of quadratic harness

of the Askey-Wilson type. It will be also shown how this new representation can be used to study different properties of ASEP's. In particular, we will provide a relatively simple proof of LDP (large deviation principle) for average overall site occupancy as well as new formulas for so called density profiles. Finally we will show how the representation can be used to study asymptotic properties of ASEP's when N , the number of sites, tends to infinity. The talk is based on joint reserach with Wlodek Bryc (University of Cincinnati).