A 3D Ginibre point field

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Abstract

We introduce a three-dimensional determinantal random point field using the concept of the quaternion determinant. Orthogonal polynomials on the space of pure quaternions are defined, and used to construct a kernel function similar to the Ginibre kernel. We find explicit formulas for the polynomials and the kernel, and calculate their asymptotics in the bulk and at the center of coordinates.

In more detail, let us define

$$\mathcal{K}_n(z,w) = (2\pi)^{-3/2} e^{-(|z|^2+|w|^2)/4} \sum_{k=0}^{n-1} \frac{P_k(z)\overline{P_k(w)}}{h_k}. \tag{1}$$

Here z and w are purely imaginary quaternions, which we identify with points in $\mathbb{R}^3$, and polynomials $P_k(z)$ are the monic quaternion polynomials of degree k, which are orthogonal with respect to the standard Gaussian measure on $\mathbb{R}^3$ and have norm $\sqrt{h_k}$.

We prove that for pure quaternions u, $P_n(us) = u^n Q_n(s)$, where $Q_n(s) \in \mathbb{Z}[s]$. In terms of these polynomials we derive a Christoffel-Darboux expression for the kernel.

Using kernel $\mathcal{K}_n(z,w)$ and the Moore-Dyson determinant for quaternion matrices, we define a random point field on $\mathbb{R}^3$ and show that (i) with high probability this field is contained in the ball of radius $2\sqrt{n}$; (ii) the average density of points in the ball is $1/\sqrt{n}$, and (iii) after rescaling the density has the form given by the function

$$\rho(s) := \frac{1}{(2\pi)^2} \frac{\sqrt{1-s^2}}{s^2} \text{ for } s \in (0,1). \tag{2}$$

We find that after suitable rescaling the correlation function for points σu and τv in the bulk of the point process is given by the formula

$$\frac{\sin(\tau-\sigma)}{\tau-\sigma} \frac{1-uv}{2}. \tag{3}$$

We also derive a scaling limit for the correlations at the center of coordinates.