Exceptional sets in nonhyperbolic contexts

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Some history of exceptional sets

$x \in [0,1)$ badly approximable if $\exists C(x) > 0$ s.t. $|\frac{p}{q} - x| > \frac{C(x)}{q^2}$.

Theorem ([Jarnik'29], [Besicovitch])

$$\dim_{\mathrm{H}}(\{x \in [0,1) \textit{ badly approximable}\}) = 1.$$

x badly approximable $\Leftrightarrow 0 \notin \overline{\mathcal{O}_f^+(x)}$, where f is the Gauss map $x \mapsto \frac{1}{x} - \left[\frac{1}{x}\right]$

(X,d) *compact metric space,* $f : X \to X$ *continuous,* $A \subset X$.

$$E_f^+(A) := \{x \in X : \overline{\mathcal{O}_f^+(x)} \cap A = \emptyset\} \qquad \textit{(forward) A-exceptional set}$$
$$I_f^+(A) := \{x \in X : \omega_f(x) \cap A = \emptyset\} \quad \textit{(forward) limit A-exceptional set}$$

$$I_f^+(A) = E_f^+(A) \cup \bigcup_{n \geq 0} f^{-n}(\{a \in A : \omega_f(a) \cap A = \emptyset\})$$

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- f *expanding* rational map of $\overline{\mathbb{C}}$, $J = J(f)$ Julia set. If either
  - $A \subset J$ "finite", or [Abercrombie–Nair 97, 06]
  - $A \subset J$, $\dim_{\mathrm{H}}(A) < \dim_{\mathrm{H}}(J)$ then [Dolgopyat 97]

$$\dim_{\mathrm{H}}(E^+_{f|J}(A)) = \dim_{\mathrm{H}}(I^+_{f|J}(A)) = \dim_{\mathrm{H}}(J)$$

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A "small" $\Rightarrow$ $E_f^+(A)$ and $I_f^+(A)$ "big" (in terms of entropy and dimension)
(This is genuinely hyperbolic phenomenon.)

Some properties of exceptional sets

Theorem (Dolgopyat 97)

For SFT $\sigma : \Sigma^+ \to \Sigma^+$ *and* $B \subset \Sigma^+ = \{0, \dots, n\}^{\mathbb{N}}$

$$h(\sigma, B) < h(\sigma) \quad \Rightarrow \quad h(\sigma, E_\sigma^+(B)) = h(\sigma, I_\sigma^+(B)) = h(\sigma).$$

- (Y, g), (X, f) conjugate via $\pi : Y \to X$ and $A \subset X$, then

$$\pi\left(I_{g|Y}^+(\pi^{-1}(A))\right) = I_f^+(A).$$

- if $W = f(W) \subset X$ compact and $A \subset X$, then $I_{f|W}^+(A \cap W) \subset I_f^+(A)$, and if $f|_W$ conjugate to SFT and $h(f, A) < h(f, W)$, then

$$h(f, E_{f|W}^+(A \cap W)) = h(f, I_{f|W}^+(A \cap W)) = h(f, W).$$

Small entropy implies full entropy

Theorem

If there exists $(W_\varepsilon)_\varepsilon \subset X$ *with* $f|_{W_\varepsilon}$ *conjugate to SFT and* $h(f, W_\varepsilon) \nearrow h(f)$, *then*

$$h(f, A) < h(f) \quad \Rightarrow \quad h(f, E_f^+(A)) = h(f, I_f^+(A)) = h(f).$$

Proof.

Eventually $h(f, A \cap W_\varepsilon) \leq h(f, A) < h(f, W_\varepsilon) \leq h(f)$. Hence

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Now take $\varepsilon \to 0$. □

Examples:

- rational map $f : J(f) \to J(f)$ degree ≥ 2 on Julia set $J(f) \subset \overline{\mathbb{C}}$
- positive entropy $C^{1+\varepsilon}$ diffeomorphism $f : X \to X$ on locally maximal invariant set $X \subset M$, M Riemannian surface

Small dimension implies full dimension?

$f : X \to X$ $C^{1+\alpha}$ conformal map, $X \subset M$, $\chi > 0, \varepsilon \in (0, \chi)$.

We call $W \subset X$ a (χ, ε)-repeller if $\sup_{x \in W} \limsup_{n \to \infty} \left| \frac{1}{n} \log |(f^n)'(x)| - \chi \right| < \varepsilon$.

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Proposition (Dolgopyat 97, Manning 81, Campos-G 16)

For any expanding (χ, ε)-repeller $W \subset X$ and $Z \subset W$ we have

$$\frac{h(f, Z)}{\chi + \varepsilon} \leq \dim_{\mathrm{H}}(Z) \leq \frac{h(f, Z)}{\chi - \varepsilon}.$$

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Proposition (Katok 95, Przytycki-Urbański 10, G 16, Campos-G 16)

If X is locally maximal, then $\exists (\mu_\varepsilon)_\varepsilon$ ergodic, $\exists (W_\varepsilon)_\varepsilon \subset X$, $(\chi(\mu_\varepsilon), \varepsilon)$-repellers:

$$\lim_{\varepsilon \to 0} \dim_{\mathrm{H}}(W_\varepsilon) = \mathrm{DD}(f) \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\chi(\mu_\varepsilon) - \varepsilon}{\chi(\mu_\varepsilon) + \varepsilon} = 1,$$

$\mathrm{DD}(f)$ *being the dynamical dimension of f.*

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$f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ rational map, $X = J = J(f)$ Julia set, $f = f|_J$, $A \subset J$

$$\mathrm{DD}(f) := \sup\{\dim_{\mathrm{H}} \mu = \frac{h(\mu)}{\chi(\mu)} : h(\mu) > 0\} \quad \text{dynamical dimension}$$

$$\mathrm{DD}(f) = \mathrm{hD}(f) := \sup_{Y \text{ conf. rep.}} \dim_{\mathrm{H}}(Y) \quad \text{hyperbolic dimension} \qquad \text{[Przytycki–Urbański 10]}$$

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□

Some more history of exceptional sets

$f : \mathbb{T}^n \to \mathbb{T}^n$ hyperbolic automorphism

Study the nature of (general) compact invariant sets.

- [Smale] $\nexists$ one-dimensional compact invariant set $\subset \mathbb{T}^2$
- Given curve $\gamma : I \to \mathbb{T}^n$, exists $x \in \gamma(I)$ with dense orbit if γ is
  - C^2 [Franks 77], or
  - rectifiable [Mañé 79],
  - but this does not extend to merely continuous curves [Hancock 78].
- [Bowen 78, Przytycki 80, Przytycki–Urbański 89]

$\Rightarrow$ Exceptional sets cannot contain any rectifiable path

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$f : X \to X$ $C^{1+\varepsilon}$ diffeomorphism, $X \subset M$ locally maximal, M Riemannian surface

μ ergodic with positive entropy: $\dim_{\mathrm{H}} \mu = \frac{h(\mu)}{\chi^+(\mu)} + \frac{h(\mu)}{|\chi^-(\mu)|}$ **[Young 82]**

Theorem ([Campos-G 18])

$$\dim_{\mathrm{H}}(A) < \dim_{\mathrm{H}} \mu \quad \Rightarrow \quad h(f, I_f^+(A)) \geq h(\mu) \text{ and } \dim_{\mathrm{H}}(I_f^+(A)) \geq \dim_{\mathrm{H}} \mu$$

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There exist improved bounds provided that there is *a priori* some more information about a "hyperbolic structure" of X.

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μ positive entropy and hence hyperbolic with Lyapunov exponents $\chi^-(\mu) < 0 < \chi^+(\mu)$,
$\dim_{\mathrm{H}}(A) < \dim_{\mathrm{H}} \mu$

Proof: $\exists (W_\varepsilon)_\varepsilon \subset X$ "almost homogeneous" $(\chi^\pm(\mu), \varepsilon)$-horseshoes

$$\lim_{\varepsilon \to 0} \dim_{\mathrm{H}}(W_\varepsilon) = \frac{h(\mu)}{\chi^+(\mu)} + \frac{h(\mu)}{|\chi^-(\mu)|}, \quad \dim_{\mathrm{H}}(W_\varepsilon) = d^{\mathrm{u}}(W_\varepsilon) + d^{\mathrm{s}}(W_\varepsilon)$$

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- Marstrand-like estimates on (subsets of) product sets

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$\mathrm{DD}(f) := \sup\{\dim_{\mathrm{H}} \mu : h(\mu) > 0\}$ dynamical dimension

Theorem ([Campos-G 18])

$$\dim_{\mathrm{H}}(A) < \dim_{\mathrm{H}} \mu \quad \Rightarrow \quad \dim_{\mathrm{H}}(I_f^+(A)) \geq \dim_{\mathrm{H}} \mu$$
$$\dim_{\mathrm{H}}(A) < \mathrm{DD}(f) \quad \Rightarrow \quad \dim_{\mathrm{H}}(E_f^+(A)) = \dim_{\mathrm{H}}(I_f^+(A)) \geq \mathrm{DD}(f)$$

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Theorem ([Campos-G 18])

If $X \subset M$ is basic

$$\dim_{\mathrm{H}}(A) < \dim_{\mathrm{H}} \mu \quad \Rightarrow \quad \dim_{\mathrm{H}}(I_f^+(A)) \geq d^{\mathrm{s}}(X) + \frac{h(\mu)}{\chi^+(\mu)}$$

If $X = M$ and f Anosov

$$\dim_{\mathrm{H}}(A) < \dim_{\mathrm{H}} \mu_{\mathrm{SRB}} \quad \Rightarrow \quad \dim_{\mathrm{H}}(I_f^+(A)) \geq 1 + \frac{h(\mu_{\mathrm{SRB}})}{\chi^+(\mu_{\mathrm{SRB}})} = 2.$$

Parabéns – Herzlichen Glückwunsch – All the best – Wszystkiego najlepszego, Mariusz!

Bedlewo 2013