DIMENSION THEORY OF DYNAMICALLY DEFINED FRACTAL SETS

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Abstract: The objects of our study are fractals sets in $\mathbb{R}^d$ that are obtained as attractors or repellers of dynamical systems. The simplest case is when we consider a self similar Iterated Function Systems (IFS) (see Figure 1). This is a finite list of contractive similarity maps of $\mathbb{R}^d$. Then we apply all possible n-fold iterates of these maps on a sufficiently large closed ball. Observe that this results in a nested sequence of compact sets if all of the maps of the IFS sends this ball into itself. Whatever remains after infinitely steps it is the attractor of the self-similar IFS. If we replace the contracting similarities with more general maps like contracting conformal or affine maps (see Figure 2), then we get the self-conformal and self-affine sets in the same way. Most commonly, these so-called self-similar, self-affine or self-conformal sets have zero Lebesgue measure. So, their size can be measured by their fractal dimensions. Our goal is, on the one hand, to compute the different fractal dimensions of these attractors and we would like to find connections between the dimension of these invariant sets and other invariants of the given dynamical system.

As an example, consider one of the most well-known fractals, the Sierpiński carpet. These is the attractor of the self-similar IFS:

$$\mathcal{S} := \left\{ S_i(x) := \frac{1}{3} \cdot x + t_i \right\}_{i=1}^{8} \tag{1}$$

where $\{t_i\}_{i=1}^8$ are the 8 elements of $\left\{(\frac{k}{3}, \frac{\ell}{3}) : (k, \ell) \in \{0, 1, 2\} \times \{0, 1, 2\} \setminus \{1, 1\}\right\}$.

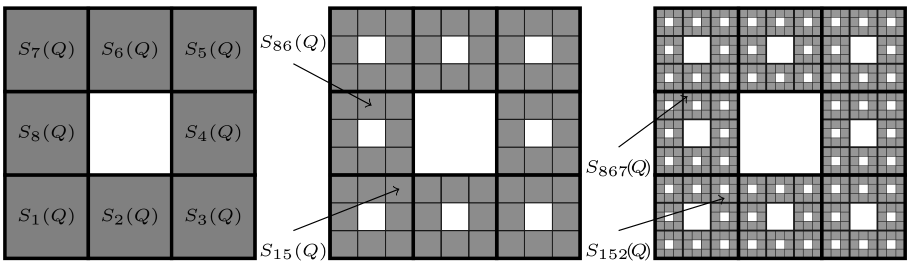

FIGURE 1. The first, second and third approximations of the Sierpiński carpet

Observe that $S_i(Q) \subset Q$ for each i, where $Q = [0, 1]^2$. The collection of little squares $S_i(Q)$, $i = 1, \ldots, 8$ on the left-hand side of Figure 1 are called first cylinders. Their union (the grey region) is the first approximation of the Sierpiński carpet. We denote it by $\Lambda_1 := \bigcup_{i=1}^8 S_i(Q)$. Similarly, the second cylinders: $\{S_i \circ S_j([Q])\}_{i,j=1}^8$ and their union, the second approximation of the Sierpiński carpet $\Lambda_2 := \bigcup_{i,j=1}^8 S_i \circ S_j(Q)$ appear in the middle in Figure 1. We define the n-th approximation of the Sierpiński carpet similarly. As the figure indicates, $\{\Lambda_i\}_{i=1}^\infty$ is a nested sequence of compact sets. Their intersection is the Sierpiński carpet

$$\Lambda := \bigcap_{i=1}^{\infty} \Lambda_n = \bigcap_{i=1}^{\infty} \bigcup_{i_1 \ldots i_n} S_{i_1 \ldots i_n}(Q), \tag{2}$$

where we use the shorthand notation $S_{i_1 \ldots i_n} := S_{i_1} \circ \cdots \circ S_{i_n}$. In this case, we have 8^n level n cylinder squares of size $3^{-n} \times 3^{-n}$. So, the Lebesgue measure of the n-th approximation Λ_n is

$$8^n \cdot 3^{-2n} \to 0. \tag{3}$$

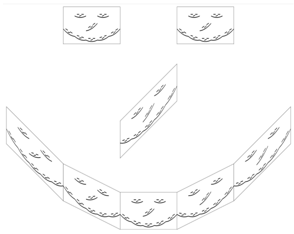

FIGURE 2. A self-affine IFS

This indicates that the dimension of the Sierpiński carpet Λ is smaller than 2. Namely, the (fractal) dimension of Λ is the number s for which it holds that if we replace 2 with s in the exponent in formula (3) that is we consider $8^n \cdot 3^{-s \cdot n}$ then this sequence converges neither to infinity nor to zero. Hence the dimension of the Sierpiński carpet is $s = \frac{\log 8}{\log 3}$. In fractal geometry, (as opposed to natural sciences) we do not really use the term fractal dimension since there are various notions of fractal dimensions. The most popular ones are the Hausdorff and the box dimensions (for self-similar sets they always coincide but for self-affine sets, they may be different).

In the three talks of my mini-course, I will introduce some of the most important notions of fractal dimensions, and we will learn the most basic theorems of the dimension theory of self-similar, self-affine and self-conformal sets.