Entropy and drift for Gibbs measures on geometrically finite manifolds

In honor of Mariusz Urbański - Bedlewo

Giulio Tiozzo
University of Toronto

April 9, 2019

Summary

1. Boundary measures
2. History and questions
3. Gibbs measures

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joint with Ilya Gekhtman

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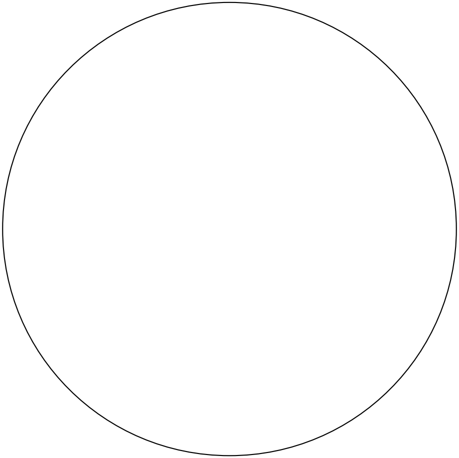

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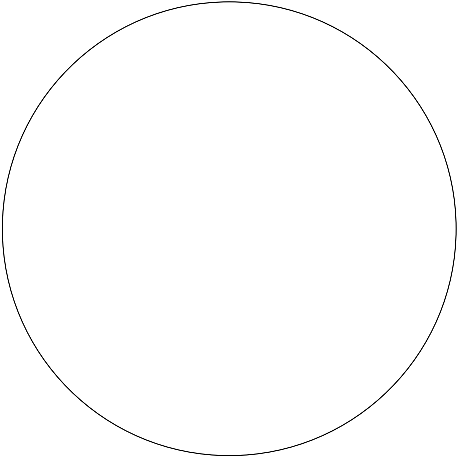

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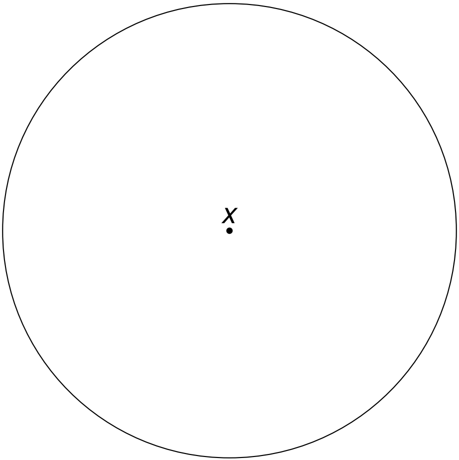

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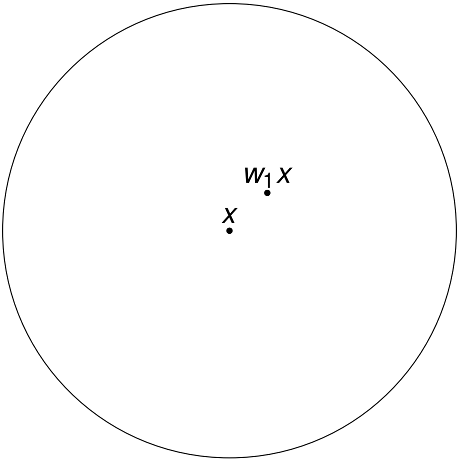

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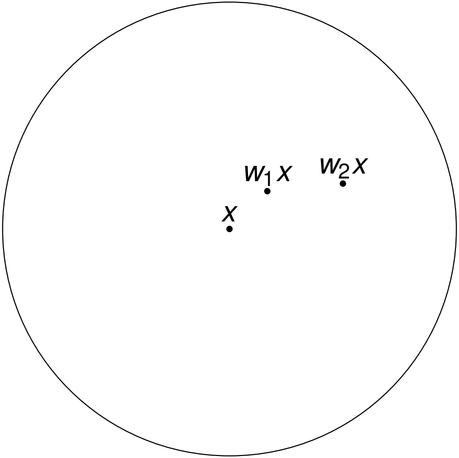

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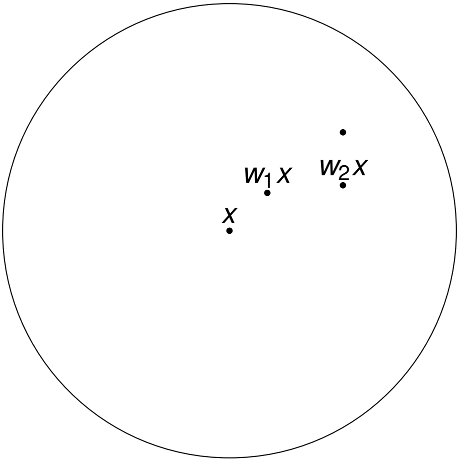

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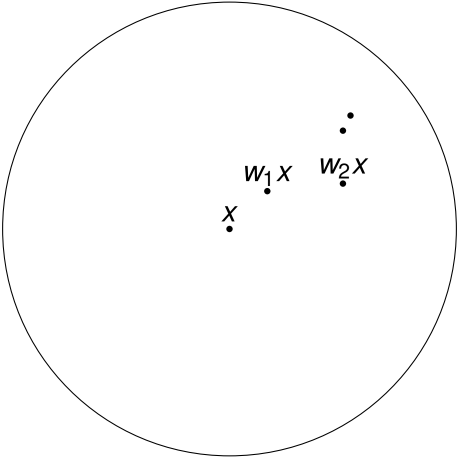

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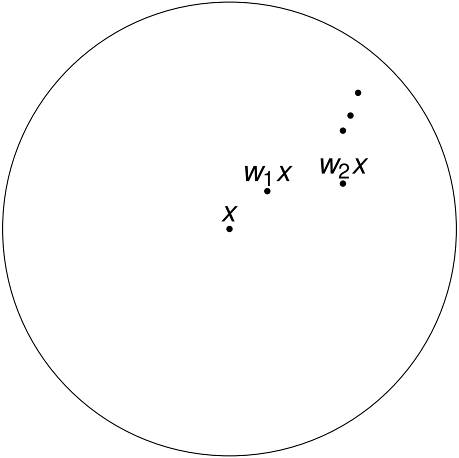

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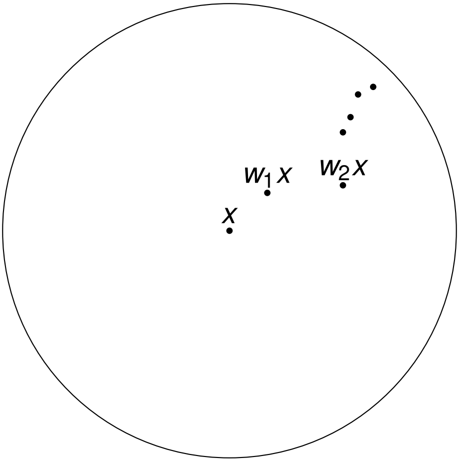

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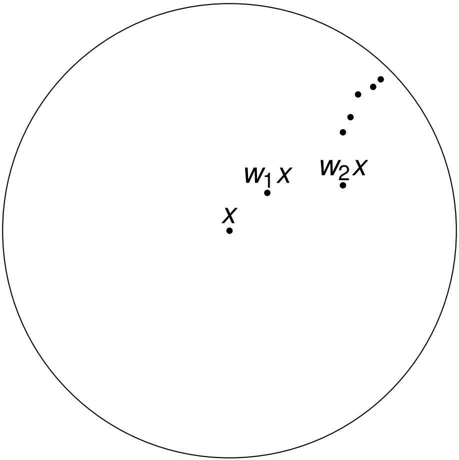

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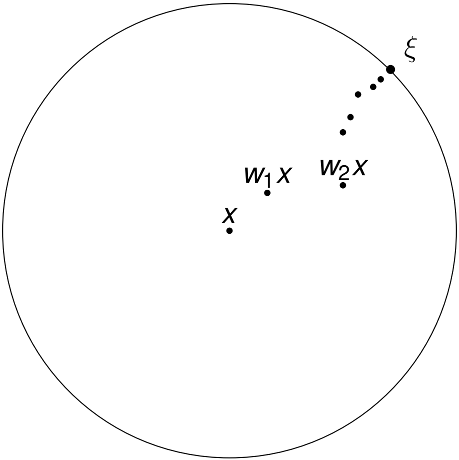

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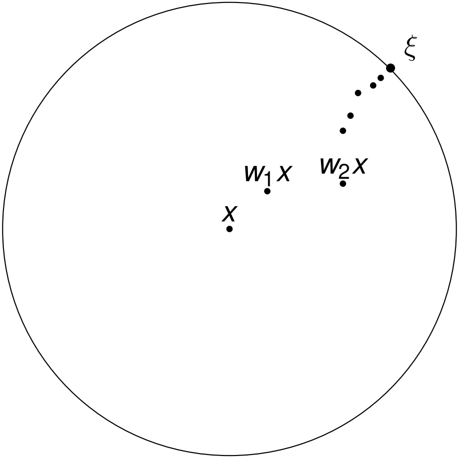

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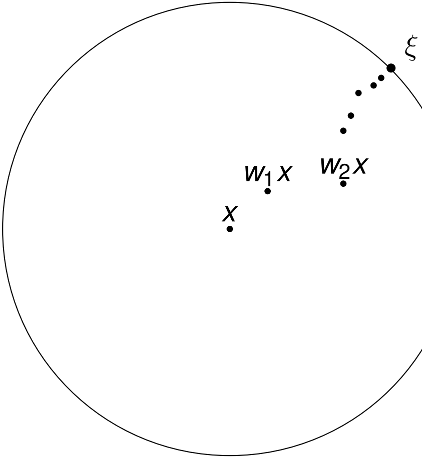

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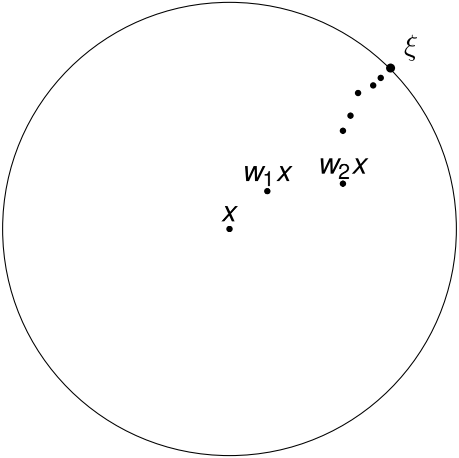

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Question. Can these two measures be the same, or in the same measure class?

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  What about in variable negative curvature? What about in higher dimensions?

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where the limit exists a.s. and is constant.

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Compare **Ledrappier** ('90): for the Brownian motion on a surface, equality holds if and only if the curvature is constant.

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(In variable negative curvature, it is not Patterson-Sullivan)

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and a measure m_F is an equilibrium state or Gibbs measure if it realizes the maximum. Moreover,

$$T^1 X \cong (\partial X \times \partial X \setminus \Delta) \times \mathbb{R}$$

hence m_F induces a Gibbs density κ_F on ∂X.

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where the integral is taken along the geodesic from x to y. Moreover, the F-ake drift is

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(In fact, result holds for $X = CAT(-1)$ space)

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Happy birthday Mariusz!!