Embeddings and Mapppings into Euclidean Spaces and Products of Continua

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1. **Unstable intersection in Euclidean spaces.**

The well known Menger-Nöbeling Theorem says that any (continuous) map from an n–dimensional compactum (i.e., a compact metric space) X into $\mathbb{R}^{2n+1}$ can be approximated arbitrarily closely by embeddings. In the classical case that X is an n-dimensional polyhedron (or manifold) the dimension $2n+1$ of the Euclidean space is optimal. However, for the Boltyanski compacta (i.e., satisfying $\dim(X \times X) < 2\dim X$) the dimension of the Euclidean space can be lowered. In fact we have (see [1], [11], [16], [3]):

$(*)$ *Any map from a compactum X into $\mathbb{R}^{2n}$ can be approximated arbitrarily closely by embeddings if and only if* $\dim(X \times X) < 2n$.

Compacta X and Y have *unstable intersection* in $\mathbb{R}^n$ if each pair of maps $X \to \mathbb{R}^n$ and $Y \to \mathbb{R}^n$ can be approximated arbitrarily closely by maps with disjoint images. Any map from a compactum X to $\mathbb{R}^n$ can be approximated by embeddings if and only if X has unstable intersection with itself in $\mathbb{R}^n$ So, $(*)$ motivated the following:

Unstable Intersection Conjecture. *Compacta X and Y have unstable intersection in $\mathbb{R}^n$ if and only if* $\dim X \times Y < n$.

The above conjecture holds in all cases except: $n = 5$, $\dim X = \dim Y = 3$ and $\dim X \times Y = 4$, in which it is open (cf., [17], [2], [4], [5], [13]).

2. **Embeddings into Euclidean spaces**

In 1933, E. R. van Kampen gave description of a certain $\mathbb{Z}/2\mathbb{Z}$-equivariant $2n$-dimensional cohomology class of the deleted product of an n-dimensional polyhedron K, which vanishes if and only if K is embeddable in $\mathbb{R}^{2n}$, provided $n \geq 3$. Many details were clarified by A. Shapiro and W. T. Wu.

If $f : X \to \mathbb{R}^m$ is an embedding then the map f^* from X^* into the sphere S^{m-1} defined by

$$f^*(x,y) = \frac{f(x) - f(y)}{||f(x) - f(y)||} \text{ for each } (x,y) \in X^* ,$$

is an equivariant map (with respect to the standard actions of $\mathbb{Z}/2\mathbb{Z}$ on X^* and S^{m-1}).

C. Weber proved a converse to this when X is a polyhedron and the dimensions are in the metastable range, which extends van Kampen's result to a wider range of dimensions and also generalizes an earlier theorem of A. Haefliger on embeddings of differentiable manifolds.

Theorem ([18]). *Let K be an $n-$dimensional compact polyhedron and m an integer such that $2m \geq 3(n+1)$. If there exist an equivariant map $F : K^* \to S^{m-1}$ then there exists a PL-embedding $f : K \to \mathbb{R}^m$ such that f^* is equivariantly homotopic to F.*

We discuss some results (e.g., [14], [6], [15]) showing that the Weber's theorem can not be extended beyond the metastable range.

3. **Embeddings into product of curves**

In 1958 J. Nagata proved the following remarkable modification of the Menger-Nöbeling theorem: *Every n-dimensional metric space, $n \geq 2$, embeds in a product $X \times \ldots \times X_{n+1}$ of 1-dimensional metric spaces.*

In 1975 K. Borsuk, answering a question of Nagata in the negative, proved that the n-sphere S^n, $n \geq 2$, does not embed in a product of n curves (1-dimensional connected compacta).

Several results dealing with embeddability of n-dimensional compacta in products of n 1-dimensional compacta where obtained by Kuperberg, Cauty, Gillman, Matveev, Rolfsen, Pol, Dydak and Koyama.

We discuss the following results proved in [8], [9], [12] and [10]. (Below, H denotes the Čech cohomology functor with the integer coefficients.)

Let X be a closed subset of the product $Y_1 \times \cdots \times Y_n$ of n curves with $H^n(X) \neq 0$. Then there is an algebraically essential map from X into the n-torus $\mathbb{T}^n$. Consequently, there exist elements $a_1, \cdots a_n \in H^1(X)$ whose cup product $a_1 \cdots a_n \in H^n(X)$ is non-zero. Such elements are linearly independent, hence $\operatorname{rank} H^1(X) \geq n$. *Moreover,* $\operatorname{cat} X > n$.

An analogous result holds for symmetric products of curves. It follows that no S^n, $n \geq 2$, can be embedded in the nth symmetric product of a curve, which gives the negative answer to a question posed by Illanes and Nadler. This result also implies that neither the projective plane P^2 nor the Klein bottle K can be embedded in the second symmetric product of a curve.

We distinguish and study some n-dimensional compacta (such as weak n-manifolds) with respect to embeddability into products of n curves. We show that if X is a locally connected weak n-manifold lying in a product of n curves then $\operatorname{rank} H^1(X) \geq n$. If $\operatorname{rank} H^1(X) = n$ then X is an n-torus. If $\operatorname{rank} H^1(X) < \infty$, then X is a polyhedron.

It follows that certain 2-dimensional compact contractible polyhedra (like "dunce hat" or "Bing house") are not embeddable in products of two curves. On the other hand, any collapsible 2-dimensional polyhedron embeds in a product of two trees. We answer a question of Cauty proving that closed surfaces embeddable in a product of two curves embed in a product of two graphs. We construct a 2-dimensional polyhedron that embeds in a product of two curves but does not embed in a product of two graphes. This solves in the negative another problem of Cauty.

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