HEAT KERNEL BOUNDS FOR SYMMETRIC MARKOV PROCESSES WITH SINGULAR JUMP MEASURES

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In the Euclidean space $\mathbb{R}^d$, we study symmetric Markov jump processes that correspond to nonlocal Dirichlet forms with singular measures. The model example that we have in mind is given by a process $Z = (Z^1, \dots, Z^d)$ of d independent symmetric one-dimensional stable processes. We consider Dirichlet forms that generate pure jump process X whose jump intensity is comparable to the one of Z. The aim of this talk is to prove sharp upper and lower heat kernel bounds for X in terms of the heat kernel of Z. Finally, we present a conjecture on heat kernel bounds for a general class of symmetric Markov jump processes.

Here are some more details. Let $\alpha \in (0,2)$. Assume $J, J^\alpha : \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag} \to \mathbb{R}$ are two non-negative functions. We assume $J^\alpha(x,y) = |y^i - x^i|^{-1-\alpha}$ if $x^i \neq y^i$ for some i and $x^j = y^j$ for every $j \neq i$. Furthermore, we assume

$$\Lambda^{-1} J^\alpha(x,y) \leq J(x,y) \leq \Lambda J^\alpha(x,y) \qquad (x \neq y)\,. \tag{1}$$

Set

$$D = \{u \in L^2(\mathbb{R}^d) |\, \mathcal{E}(u,u) < \infty\}$$

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^d} \Big(\sum_{i=1}^d \int_{\mathbb{R}} \big(u(x+e^i\tau) - u(x)\big)\big(v(x+e^i\tau) - v(x)\big) J(x, x+e^i\tau) d\tau \Big) dx\,,$$

$(\mathcal{E}, \mathcal{F})$ is a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d)$ where $\mathcal{F} = \overline{C_c^1(\mathbb{R}^d)}^{\mathcal{E}_1}$. It is proved in [1] that there is a corresponding conservative Hunt process X together with a Hölder continuous transition density p_t. Furthermore, [1] proves sharp lower pointwise bounds and some upper pointwise bounds on p_t. Here, we prove sharp upper bounds.

Theorem 1 (Kim/MK/Kumagai). *There exists $C \geq 1$ such that for all $t > 0, x, y \in \mathbb{R}^d$*

$$C^{-1} t^{-d/\alpha} \prod_{i=1}^d \left(1 \wedge \frac{t^{1/\alpha}}{|x^i - y^i|} \right)^{1+\alpha} \leq p_t(x,y) \leq C t^{-d/\alpha} \prod_{i=1}^d \left(1 \wedge \frac{t^{1/\alpha}}{|x^i - y^i|} \right)^{1+\alpha}.$$

The constant C depends on d, α and Λ but, other than that, is independent of J.

This result should be contrasted with the main result of [2]. We conjecture the following more general result:

Conjecture: Let Z be a non-degenerate α-stable process in $\mathbb{R}^d$. Let X be a symmetric Markov process whose Dirichlet form has a jump intensity $J(x,dy)$ that is comparable to the one of Z. Then the heat kernel of X is comparable to the one of Z.

References

[1] Fangjun Xu, *A class of singular symmetric Markov processes.* Potential Anal. **38** (1), pp. 207–232, 2013

[2] Zhen-Qing Chen and Takashi Kumagai, *Heat kernel estimates for stable-like processes on d-sets.* Stochastic Process. Appl. **108** (1), pp. 27–62, 2003