BROWNIAN MOTION WITH GENERAL DRIFT

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We consider the problem of constructing a (unique) weak solution to the SDE

$$dX(t) = -b(X(t))dt + \sqrt{2}dW(t), \quad X(0) = x \in \mathbb{R}^d, \tag{1}$$

where $W(t)$ is a d-dimensional Brownian motion, $d \geq 3$, with drift $b : \mathbb{R}^d \to \mathbb{R}^d$ in the class of *weakly* form-bounded vector fields, i.e. $|b| \in L^1_{\rm loc}$ and there exist constants $\delta > 0$ and $\lambda = \lambda_\delta > 0$ such that

$$\||b|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}}\|_{L^2 \to L^2} \leq \sqrt{\delta},$$

(write $b \in \mathbf{F}_\delta^{1/2}$).

The class $\mathbf{F}_\delta^{1/2}$ contains as proper subclasses $[L^p + L^\infty]^d$, $p > d$ (by Hölder's inequality) and $[L^d + L^\infty]^d$ (by Sobolev's inequality), with relative bound δ that can be chosen arbitrarily small. $\mathbf{F}_\delta^{1/2}$ also contains vector fields having critical-order singularities, such as $b(x) = \frac{d-2}{2}\delta|x|^{-2}x$ (by Hardy's inequality) or, more generally, vector fields in the weak L^d class (by Strichartz' inequality), the Campanato-Morrey class or the Chang-Wilson-Wolff class, with δ depending on the respective norm of the vector field in these classes. The class $\mathbf{F}_\delta^{1/2}$ also contains as a proper subclass the Kato class $\mathbf{K}_\delta^{d+1} = \{|b| \in L^1_{\rm loc} : \||b|(\lambda - \Delta)^{-\frac{1}{2}}\|_{L^1 \to L^1} \leq \delta\}$ (e.g. by interpolation).

The first principal result on the problem of constructing a (unique in law) weak solution to the SDE (1) with a locally unbounded *general* b is due to N. I. Portenko [1]: if $|b| \in L^p + L^\infty$, $p > d$, then there exists a unique in law weak solution to (1). This result has been strengthened in [2] for b in $\mathbf{K}_0^{d+1} := \bigcap_{\delta > 0} \mathbf{K}_\delta^{d+1}$.

We prove that the SDE (1) with $b \in \mathbf{F}_\delta^{1/2}$,

$$\delta < m_d^{-1}\frac{4(d-2)}{(d-1)^2}, \qquad m_d := \pi^{\frac{1}{2}}(2e)^{-\frac{1}{2}}d^{\frac{d}{2}}(d-1)^{\frac{1-d}{2}},$$

has a weak solution for every $x \in \mathbb{R}^d$. (Thus, the solvability of (1) depends on the value of the relative bound δ, as, the example of $b(x) = \frac{d-2}{2}\delta|x|^{-2}x$ shows, it should.) The weak solutions of (1) constitute a Feller semigroup on C_∞, the starting object in our approach. The weak solution to (1) is unique among all weak solutions that can be constructed via "reasonable" approximations of b, i.e. the ones that keep the value of the relative bound δ intact.

We note that the class $\mathbf{F}_\delta^{1/2}$ destroys the two-sided Gaussian bounds on the fundamental solution of $\partial_t + \Lambda(b)$, $\Lambda(b) \supset -\Delta + b\cdot\nabla$ (e.g. consider $b(x) = \pm\frac{d-2}{2}\delta|x|^{-2}x$; this is in contrast to the Kato class $\mathbf{K}_\delta^{d+1}$, which preserves the Gaussian bounds for $\delta > 0$ small). Instead, we appeal to the weighted estimates on the resolvent of $\Lambda(b)$ in L^p, $p > d-1$.

This is joint work with Yu.A.Semënov (Toronto).

References

[1] N. I. Portenko. Generalized Diffusion Processes. *AMS*, 1990.
[2] R. Bass, Z.-Q. Chen. Brownian motion with singular drift. *Ann. Prob.*, 31 (2003), p. 791-817.
[3] D. Kinzebulatov, Yu.A. Semënov. Brownian motion with general drift, *Preprint*, arXiv:1710.06729, 10 p (2017).