SCHAUDER ESTIMATES FOR NON-LOCAL OPERATORS

FRANZISKA KÜHN

Let $(X_t)_{t\geq 0}$ be a Lévy process with infinitesimal generator $(A, \mathcal{D}(A))$. By the Lévy–Khintchine formula, A has a representation of the form

$$Af(x) = b\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}\left(Q \cdot \nabla^2 f(x)\right) + \int_{y\neq 0} \left(f(x+y) - f(x) - y\cdot\nabla f(x)1_{(0,1)}(|y|)\right)\,\nu(dy)$$

for any smooth function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support. In this talk we study the regularity of solutions f to the Poisson equation

$$Af = g.$$

We are interested in the following question: If $f \in \mathcal{D}(A)$ is such that $Af = g$ has a certain regularity (say, Hölder continuous), then what does this tell us about the regularity of f? We will show that gradient estimates for the transition density p_t of $(X_t)_{t\geq 0}$ can be used to establish Schauder estimates for f. More precisely, if

$$\int_{\mathbb{R}^d} |\nabla p_t(x)|\,dx \leq ct^{-1/\alpha}, \qquad t \in (0,1),$$

for some constants $c > 0$ and $\alpha \in (0,2]$, then this implies that the Schauder estimate

$$\|f\|_{\mathcal{C}_b^{\alpha+\delta}(\mathbb{R}^d)} \leq K\left(\|Af\|_{\mathcal{C}_b^{\delta}(\mathbb{R}^d)} + \|f\|_\infty\right)$$

holds for some finite constant $K > 0$; in particular,

$$Af = g \in \mathcal{C}_b^{\delta}(\mathbb{R}^d) \implies f \in \mathcal{C}_b^{\alpha+\delta}(\mathbb{R}^d)$$

for all $\delta \geq 0$. Here $\mathcal{C}_b^{\lambda}(\mathbb{R}^d)$, $\lambda \geq 0$, denote so-called Hölder–Zygmund spaces; they coincide with the "classical" Hölder spaces $C_b^{\lambda}(\mathbb{R}^d)$ for $\lambda \notin \mathbb{N}$. Since gradient estimates have been intensively studied, our result applies to a wide class of Lévy processes, e. g. subordinate Brownian motions and stable Lévy processes. We will present several examples in this talk. As a consequence of the result, we establish for a large class of Lévy processes that the domain $\mathcal{D}(A)$ is an algebra, i. e. $f, g \in \mathcal{D}(A)$ implies $f \cdot g \in \mathcal{D}(A)$, and we obtain an explicit expression for $A(f\cdot g)$. Moreover, we will give some remarks on possible extensions to infinitesimal generators of Feller processes.

References

[1] Franziska Kühn, *Schauder estimates for elliptic equations associated with Lévy generators.* Preprint arXiv 1812.06124.

[2] Franziska Kühn, *Schauder estimates for Poisson equations associated with non-local Feller generators.* Preprint arXiv 1902.01760.