HARDY INEQUALITY AND FRACTIONAL HARDY INEQUALITY FOR DUNKL LAPLACIAN

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A general Hardy inequality is of the form

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \left(\frac{|N-p|}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx,$$

for $u \in C_0^\infty(\mathbb{R}^N)$ or $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ respectively with respect to $1 \leq p < N$ or $p > N$. It is known that the constant $\left(\frac{|N-p|}{p}\right)^p$ is sharp and never attained in the corresponding spaces $\dot{W}_p^1(\mathbb{R}^N)$ or $\dot{W}_p^1(\mathbb{R}^N \setminus \{0\})$ respectively. A remarkable work on the same is done by R.L Frank and R. Seiringer in [1]. They have proven the sharp Hardy inequality with sharp constants as follows: for $p \geq 1$, $0 < s < 1$ and $u \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \geq C_{N,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx,$$

where the constant $C_{N,s,p}$ is sharp. Our aim is to prove both Hardy and fractional Hardy inequality in Dunkl setting.

Theorem 1. *Let $1 \leq p < \infty$. Let u be a real valued $G-$ invariant function. If $u \in C_0^\infty(\mathbb{R}^N)$ if $d_k > p$ and $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $d_k < p$ then the following inequality holds:*

$$\int_{\mathbb{R}^N} |\nabla_k u(x)|^p d\mu_k(x) \geq \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} d\mu_k(x). \tag{1}$$

The constant $\left|\frac{d_k-p}{p}\right|^p$ given in the inequality is optimal.

Theorem 2. *Let $2 \leq p < \infty$. Let u be a real valued $G-$ invariant function. If $u \in C_0^\infty(\mathbb{R}^N)$ if $d_k > p$ and $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $d_k < p$ then the following inequality holds:*

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) - \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} d\mu_k(x) \geq c_p \int_{\mathbb{R}^N} \frac{|\nabla_k v|^p}{|x|^{d_k-p}} d\mu_k(x), \tag{2}$$

where $c_p := \min_{0<\tau<1/2} \left((1-\tau)^p - \tau^p + p\tau^{p-1}\right)$. When $p = 2$ the equality holds and with $c_2 = 1$.

Theorem 3. *Let $d_k \geq 1$ and $0 < s < 1$. If $u \in \dot{W}_p^s(\mathbb{R}^N)$ when $2 \leq p < d_k/s$ or $u \in \dot{W}_p^s(\mathbb{R}^{\mathrm{N}} \setminus \{0\})$ when $p > d_k/s$, the following inequality holds;*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^p \Phi(x,y) d\mu_k(x) d\mu_k(y) \geq C_{d_k,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} d\mu_k(x), \tag{3}$$

where

$$\Phi(x,y) := \frac{1}{\Gamma((d_k+ps)/2)} \int_0^\infty s^{\frac{d_k+ps}{2}-1} \tau_y^k \left(e^{-s|.|^2}\right)(x) ds \tag{4}$$

and τ_y^k denotes dunkl translation.

References

[1] R. Frank, R. Seiringer, *Non-linear ground state representations and sharp Hardy inequalities.* J. Funct. Anal. 255 (2008), 3407-3430.