SHARP HARDY'S INEQUALITY FOR LAGUERRE AND HERMITE EXPANSIONS

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Hardy and Littlewood [1] proved the following inequality for Fourier coefficients

$$\sum_{k\in\mathbb{Z}} \frac{|\hat{f}(k)|}{|k|+1} \lesssim \|f\|_{\mathrm{Re}H^1}, \tag{1}$$

where $\mathrm{Re}H^1$ denotes the real Hardy space constituted by the boundary values of the real parts of functions in the Hardy space $H^1(\mathbb{D})$, where $\mathbb{D}$ is the unit disk on the plane.

Kanjin [2] initiated investigation of analogues of (1) for orthogonal expansions. He proved the one-dimensional version of the following inequality

$$\sum_{n\in\mathbb{N}^d} \frac{|\langle f, h_n\rangle|}{(n_1+\ldots+n_d+1)^E} \lesssim \|f\|_{H^1(\mathbf{R}^d)}, \qquad f\in H^1(\mathbf{R}^d), \tag{2}$$

where $n=(n_1,\ldots,n_d)$, $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2(\mathbf{R}^d)$, $\{h_n\}_{n\in\mathbf{N}^d}$ are the Hermite functions, and $H^1(\mathbf{R}^d)$ denotes the Hardy space. We will refer to the constant E as the admissible exponent.

We introduce a method of proving multi-dimensional Hardy's inequalities associated with orthogonal expansions, which is described in [4]. Then we deduce the appropriate versions of (2) for the Hermite expansions and various Laguerre expansions (see [3, 5]). Moreover, the obtained inequalities are sharp in the sense that the admissible exponents cannot be lowered. In order to justify that we construct explicit counterexamples.

References

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[4] P. Plewa, *Sharp Hardy's type inequality for Laguerre expansions*, preprint (2018), arXiv:1810.08138.

[5] P. Plewa, *On Hardy's inequality for Hermite expansions*, preprint (2019), arXiv:1901.05663.