Exponential functionals of Markov additive processes

TECHNISCHE UNIVERSITÄT DRESDEN

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joint work in progress with
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Probability and Analysis

Overview

- Exponential functionals of Lévy processes
- From Lévy processes to MAPs
- Exponential functionals of MAPs:
  - Definition
  - An example
  - Main results and methodology
  - Discussion and more results
  - Open questions

Generalized Ornstein-Uhlenbeck processes

Let $(\xi_t, \eta_t)_{t\geq 0}$ be a bivariate Lévy process.
The generalized Ornstein-Uhlenbeck (GOU) process $(V_t)_{t\geq 0}$ driven by (ξ, η) is given by

$$V_t = e^{-\xi_t}\left(V_0 + \int_{(0,t]} e^{\xi_{s-}} d\eta_s\right), \quad t \geq 0,$$

where V_0 is a finite random variable, independent of (ξ, η).

Lévy processes

Definition: A Lévy process in $\mathbb{R}^d$ on a probability space $(\Omega, \mathcal{F}, P)$ is a stochastic process $X = (X_t)_{t\geq 0}$, $X_t : \Omega \to \mathbb{R}^d$ satisfying the following properties:

- $X_0 = 0$ a.s.
- X has independent increments, i.e. for all $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
- X has stationary increments, i.e. for all $s, t \geq 0$ it holds $X_{s+t} - X_s \overset{d}{=} X_t$.
- X has a.s. càdlàg paths, i.e. for P-a.e. $\omega \in \Omega$ the path $t \mapsto X_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.

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A Lévy process X is uniquely determined by its characteristic triplet $(\gamma_X, \sigma_X, \nu_X)$.

Exponential functionals

Theorem (Lindner, Maller '05): The GOU process

$$V_t = e^{-\xi_t}\left(V_0 + \int_0^t e^{\xi_{s-}}\,d\eta_s\right), \quad t \geq 0,$$

solving

$$dV_t = V_{t-}dU_t + dL_t,$$

with

- $\xi_t = -\log(\mathcal{E}(U)_t)$
- (U, L) (or similarly (ξ, η)) bivariate Lévy processes
- V_0 starting random variable independent of (ξ, η)

has a (nontrivial) stationary distribution if and only if the integral

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The stationary distribution is given by the law of the so-called exponential functional V_∞.

Exponential functionals

Erickson and Maller (2005) proposed necessary and sufficient conditions for convergence of

$$V_\infty := \int_0^\infty e^{-\xi_{t-}} d\eta_t.$$

Mainly one needs:

- ξ tends to infinity
- η has a finite $\log^+$-moment

Convergence of exponential functionals

Precisely, they stated

Theorem (Erickson, Maller '05):

V_∞ exists as a.s. limit as $t \to \infty$ of $\int_0^t e^{-\xi_{s-}} d\eta_s$ if and only if

$$\lim_{t\to\infty} \xi_t = \infty \text{ a.s. and} \quad I_{\xi,\eta} = \int_{(e^a,\infty)} \left(\frac{\log y}{A_\xi(\log y)} \right) |\bar{\nu}_\eta(dy)| < \infty,$$

where

$$A_\xi(x) = \gamma_\xi + \bar{\nu}_\xi^+(1) + \int_1^x \bar{\nu}_\xi^+(y)dy,$$

with

$$\bar{\nu}_\xi^+(x) = \nu_\xi((x,\infty)), \quad \bar{\nu}_\xi^-(x) = \nu_\xi((-\infty,-x)), \quad \bar{\nu}_\xi(x) = \bar{\nu}_\xi^+(x) + \bar{\nu}_\xi^-(x),$$

and $\bar{\nu}_\eta^+$, $\bar{\nu}_\eta^-$ and $\bar{\nu}_\eta$ defined likewise.

Hereby $a > 0$ is chosen such that $A_\xi(x) > 0$ for all $x > 0$ and its existence is guaranteed whenever $\lim_{t\to\infty} \xi_t = \infty$ a.s.

Convergence of exponential functionals

Further:

Theorem (Erickson, Maller '05, continued):
If $\lim_{t\to\infty} \xi_t = \infty$ a.s. but $I_{\xi,\eta} = \infty$, then

$$\left| \int_0^t e^{-\xi_{s-}} d\eta_s \right| \xrightarrow{\mathbb{P}} \infty, \tag{1}$$

while for $\lim_{t\to\infty} \xi_t = -\infty$ or oscillating ξ either (1) holds, or there exists some $k \in \mathbb{R} \setminus \{0\}$ such that

$$\int_0^t e^{-\xi_{s-}} d\eta_s = k(1 - e^{-\xi_t}) \quad \text{for all } t > 0 \quad \text{a.s.}$$

Markov additive processes (MAPs)

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- Additive component $(\xi_t, \eta_t)_{t\geq 0}$:

$$(\xi_t, \eta_t) := (X_t^{(1)}, Y_t^{(1)}) + (X_t^{(2)}, Y_t^{(2)}), \quad t \geq 0.$$

  - $(X_t^{(1)}, Y_t^{(1)})$ behaves in law like a bivariate Lévy process $(\xi_t^{(j)}, \eta_t^{(j)})$ whenever $J_t = j$,
  - $(X_t^{(2)}, Y_t^{(2)})$ is a pure jump process given by

$$(X_t^{(2)}, Y_t^{(2)}) = \sum_{n\geq 1} \sum_{i,j\in\mathcal{S}} Z_n^{(i,j)} \mathbb{1}_{\{J_{T_n-}=i, J_{T_n}=j, T_n\leq t\}},$$

  for i.i.d. random variables $Z_n^{(i,j)}$ in $\mathbb{R}^2$.

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  for i.i.d. random variables $Z_n^{(i,j)}$ in $\mathbb{R}^2$.

As starting value under $\mathbb{P}_j$ we use $(\xi_0, \eta_0, J_0) = (0, 0, j)$.
Throughout neither ξ nor η is degenerate constantly equal to 0.

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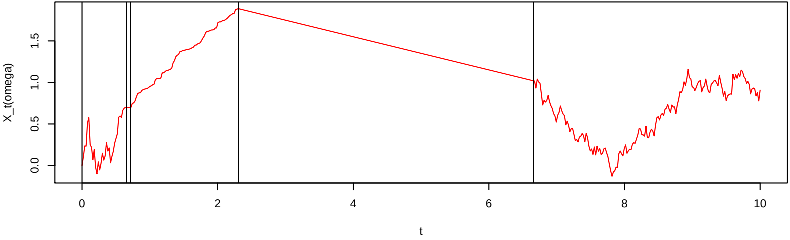
- $(X_t^{(1)}, Y_t^{(1)}) \equiv 0$: (bivariate) continuous-time Markov chain in $\mathbb{R}^2$
- $(X_t^{(2)}, Y_t^{(2)}) \equiv 0$: no common jumps of $(J_t)_{t\geq 0}$ and $(\xi_t, \eta_t)_{t\geq 0}$:



Exponential functionals of MAPs

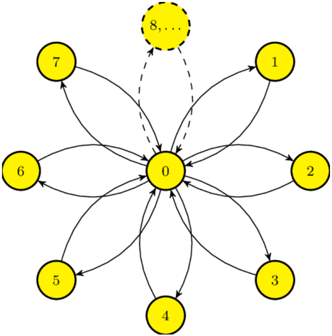
Exponential functionals of MAPs

Given a bivariate Markov additive process $(\xi_t, \eta_t, J_t)_{t\geq 0}$ with Markovian component $(J_t)_{t\geq 0}$, we denote

$$\mathfrak{E}(t) := \mathfrak{E}_{(\xi,\eta)}(t) := \int_{(0,t]} e^{-\xi_{s-}} d\eta_s, \quad 0 < t < \infty.$$

Literature:
Some recent results on $\mathfrak{E}$ for $\eta_t = t$ on arXiv
(Salminen et al./Stephenson)

An example[1]



- $\mathcal{S} = \mathbb{N}_0$
- $(J_t)_{t \geq 0}$ continuous time petal flower Markov chain with intensity matrix

$$Q = (q_{i,j})_{i,j \in \mathbb{N}_0} = \begin{pmatrix} -q & q_{0,1} & q_{0,2} & \cdots \\ q & -q & 0 & \cdots \\ q & 0 & -q & \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

for $q > 0$ fixed and $q_{0,j} = q p_{0,j}$ $j \in \mathbb{N}$.

$(J_t)_{t \geq 0}$ is irreducible, recurrent with stationary distribution

$$\pi_0 = \frac{1}{2}, \quad \text{and} \quad \pi_j = \frac{p_{0,j}}{2} = \frac{q_{0,j}}{2q}, \ j \in \mathbb{N}$$

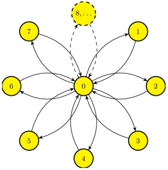
[1] Thanks to Gerold Alsmeyer for the picture!

An example

Now choose ξ and η to be conditionally independent with

$$\xi_t = X_t^{(2)} = \sum_{n\geq 1} \sum_{i,j\in\mathbb{N}_0} Z_n^{(i,j)} \mathbb{1}_{\{J_{T_n-}=i, J_{T_n}=j, T_n\leq t\}}, \quad Z_n^{(i,j)} := \begin{cases} -p_{0,j}^{-1}, & i=0, \\ 2+p_{0,i}^{-1}, & j=0, \\ 0, & \text{else.} \end{cases}$$

Then



$$\xi_{\tau_n(0)} = 2n \to \infty \quad \mathbb{P}_0\text{-a.s.}$$

$$\text{but} \quad \liminf_{t\to\infty} \xi_t = -\infty.$$

Thus ξ is oscillating.

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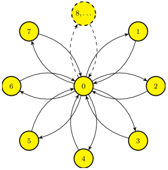

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Choosing $\eta_t = \int_{(0,t]} \gamma_{\eta^{(J_s)}} ds$ with $\gamma_{\eta^{(j)}} = \begin{cases} 1, & j=0, \\ 0, & \text{otherwise,} \end{cases}$

we observe under $\mathbb{P}_0$

$$\int_{(0,t]} e^{-\xi_{s-}} d\eta_s = \int_{(0,t]} e^{-\xi_{s-}} \gamma_{\eta^{(J_s)}} ds = \int_{(0,t]} e^{-N_{s-}} \mathbb{1}_{\{J_s=0\}} ds.$$

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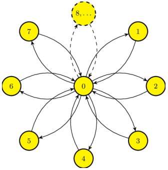

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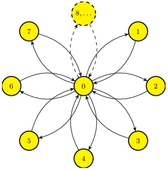

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Methodology

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Lévy case (Erickson & Maller '05): The exponential functional

$$\int_{(0,\infty)} e^{-\xi_{t-}}\, d\eta_t,$$

can be discretized for any $h > 0$ as:

$$\begin{aligned}\int_{(0,nh]} e^{-\xi_{s-}}\, d\eta_s &= \sum_{i=0}^{n-1} \int_{(ih,(i+1)h]} e^{-\xi_{s-}}\, d\eta_s \\ &= \sum_{i=0}^{n-1} \left(\prod_{j=0}^{i-1} e^{-(\xi_{(j+1)h} - \xi_{jh})} \right) \int_{(ih,(i+1)h]} e^{-(\xi_{s-} - \xi_{ih})}\, d\eta_s,\end{aligned}$$

Thus: Convergence of the integral is strongly connected to convergence of discrete-time perpetuities as studied by Goldie & Maller (2000).

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where

- $(A_n, B_n)_{n \in \mathbb{N}}$, sequence of random vectors in $\mathbb{R}^2$, modulated by
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"Modulated" in the sense that

- conditionally on $M_n = i_n \in \mathcal{S}$, $n = 0, 1, 2, \ldots$ the random vectors $(A_1, B_1), (A_2, B_2), \ldots$ are independent, and
- for all $n \in \mathbb{N}$ the conditional law of (A_n, B_n) is temporally homogeneous and depends only on $(i_{n-1}, i_n) \in \mathcal{S}^2$.

Methodology III

We shall

- find "good" discretizations of $\mathfrak{E}_{\xi,\eta}(t)$ and
- apply/extend Alsmeyer & Buckmanns results/techniques,

to obtain necessary and/or sufficient conditions for convergence of $\mathfrak{E}_{\xi,\eta}(t)$ as $t \to \infty$.

Results

Define

$$\mathbb{T}_j := \{t \geq 0 : J_t = t\},$$

and

$$\tau_1(j) := \text{first return time to } j, \qquad \tau_1^-(j) := \text{first exit time of } j.$$

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Main Theorem (B&Sideris, '19):
(i) Assume that $\lim_{t \in \mathbb{T}_j, t \to \infty} \xi_t = \infty$ for some/all $j \in \mathcal{S}$ and that

$$\int_{(1,\infty)} \frac{\log q}{\int_{(0,\log q]} \mathbb{P}_j(\xi_{\tau_1(j)} > u) du} \mathbb{P}_j \left(\sup_{0 \leq t \leq \tau_1(j)} \left| \int_0^t e^{-\xi_{s-}} d\eta_s \right| \in dq \right) < \infty,$$

for some/all $j \in \mathcal{S}$, then $\mathfrak{E}_{(\xi,\eta)}(t) \to \mathfrak{E}^{\infty}_{(\xi,\eta)}$ $\mathbb{P}_\pi$-a.s. as $t \to \infty$ for some random variable $\mathfrak{E}^{\infty}_{(\xi,\eta)}$.

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(ii) If $\lim_{t\in\mathbb{T}_j, t\to\infty} \xi_t = \infty$ for some $j \in \mathcal{S}$ and

$$\int_{(1,\infty)} \frac{\log q}{\int_{(0,\log q]} \mathbb{P}_j(\xi_{\tau_1(j)} > u)du} \mathbb{P}_j\left(\Big| \int_{(\tau_1^-(j),\tau_1(j)]} e^{-\xi_{s-}} d\eta_s + Y^{b,\eta}_{\tau_1^-(j)} \Big| \in dq \right) < \infty, \tag{2}$$

then there exists a probability measure $Q_j = \mathbb{P}_j(\mathfrak{E}^\infty_{(\xi,\eta)} \in \cdot)$ on $\mathbb{R}$, such that $\mathfrak{E}_{(\xi,\eta)}(t) \to \mathfrak{E}^\infty_{(\xi,\eta)}$ in $\mathbb{P}_j$-probability.

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(iii) If $\liminf_{t \in \mathbb{T}_j, t \to \infty} \xi_t < \infty$ for some $j \in \mathcal{S}$, then either there exists a (unique) sequence $\{c_i, i \in \mathcal{S}\}$ in $\mathbb{R}$ such that

$$\mathfrak{E}_{(\xi,\eta)}(t) = \int_{(0,t]} e^{-\xi_{s-}} d\eta_s = c_{J_0} - c_{J_t} e^{-\xi_t} \quad \mathbb{P}_\pi\text{-a.s.}$$

for all $t \geq 0$, or

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For $\mathfrak{E}_{\xi,\eta}$ we observe: $\mathfrak{E}_{\xi,\eta}$ is degenerate iff there exists a (unique) sequence $\{c_i, i \in \mathcal{S}\}$ in $\mathbb{R}$ such that

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Further:

Proposition (B&Sideris, '19+) Assume (3) holds for all $t \geq 0$ and some sequence $\{c_i, i \in \mathcal{S}\}$. Then

$$\eta_t = -\int_{(0,t]} c_{J_{s-}} dU_s - \int_{(0,t]} dc_{J_s}, \quad t \geq 0, \ \mathbb{P}_\pi\text{-a.s.} \qquad (4)$$

where $(U_t)_{t \geq 0} = (\mathrm{Log}\,(e^{-\xi_t}))_{t \geq 0}$.
Conversely, if (4) holds for some sequence $\{c_i, i \in \mathcal{S}\}$, then (3) is fulfilled for all $t \geq 0$.

Degeneracy, continued

$$\eta_t = -\int_{(0,t]} c_{J_{s-}}\, dU_s - \int_{(0,t]} dc_{J_s}, \quad t \geq 0,\ \mathbb{P}_\pi\text{-a.s.}$$

implies

Degeneracy, continued

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implies

- If $(X_t^{(2)}, Y_t^{(2)}) \equiv 0$ (that is $\Delta\xi_{T_n} = 0 = \Delta\eta_{T_n}$ $\mathbb{P}_\pi$-a.s. for all n), then $c_i \equiv c$, i.e. $\eta_t = -cU_t$ and

$$\mathfrak{E}_{(\xi,\eta)}(t) = \int_{(0,t]} e^{-\xi_{s-}}\, d\eta_s = c(1 - e^{-\xi_t}) \quad \mathbb{P}_\pi\text{-a.s. for all } t \geq 0.$$

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- If $\Delta\xi_{T_n} = 0$ $\mathbb{P}_\pi$-a.s. for all n, then $Z_\eta^{(i,j)} = c_i - c_j$.

Degeneracy, continued

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- If $\Delta\xi_{T_n} = 0$ $\mathbb{P}_\pi$-a.s. for all n, then $Z_\eta^{(i,j)} = c_i - c_j$.
- If $\Delta\eta_{T_n} = 0$ $\mathbb{P}_\pi$-a.s. for all n, then $Z_\xi^{(i,j)} = \log \frac{c_j}{c_i}$.

Although we can state

necessary and sufficient conditions

for convergence of $\mathfrak{E}_{(\xi,\eta)}(t)$, these are hardly applicable.

Splitting

To obtain easier (sufficient) conditions, we split the exponential functional in two pieces:

$$\begin{aligned}
\mathfrak{E}_{(\xi,\eta)}(t) &= \int_{(0,t]} e^{-\xi_{s-}} d(\gamma_s^\eta + W_s^\eta + Y_s^{b,\eta} + Y_s^{s,\eta} + Y_s^{(2)}) \\
&= \int_{(0,t]} e^{-\xi_{s-}} d(\underbrace{\gamma_s^\eta}_{\text{drift}} + \underbrace{W_s^\eta + Y_s^{s,\eta}}_{\text{martingale part}}) + \int_{(0,t]} e^{-\xi_{s-}} d(\underbrace{Y_s^{b,\eta} + Y_s^{(2)}}_{\text{"big" jumps}}) \\
&=: \mathfrak{E}^{(1)}(t) + \mathfrak{E}^{(2)}(t).
\end{aligned}$$

Splitting

To obtain easier (sufficient) conditions, we split the exponential functional in two pieces:

$$\begin{aligned}\mathfrak{E}_{(\xi,\eta)}(t) &= \int_{(0,t]} e^{-\xi_{s-}} d(\gamma_s^\eta + W_s^\eta + Y_s^{b,\eta} + Y_s^{s,\eta} + Y_s^{(2)}) \\ &= \int_{(0,t]} e^{-\xi_{s-}} d(\underbrace{\gamma_s^\eta}_{\text{drift}} + \underbrace{W_s^\eta + Y_s^{s,\eta}}_{\text{martingale part}}) + \int_{(0,t]} e^{-\xi_{s-}} d(\underbrace{Y_s^{b,\eta} + Y_s^{(2)}}_{\text{"big" jumps}}) \\ &=: \mathfrak{E}^{(1)}(t) + \mathfrak{E}^{(2)}(t).\end{aligned}$$

- $\mathfrak{E}^{(1)}(t)$ converges a.s. under rather weak conditions.

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To obtain easier (sufficient) conditions, we split the exponential functional in two pieces:

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- $\mathfrak{E}^{(1)}(t)$ converges a.s. under rather weak conditions.
- $\mathfrak{E}^{(2)}(t)$ can be embedded in a discrete time setting and directly corresponds to a Markov modulated perpetuity.

The characteristic κ_ξ

Recall: Lévy processes in $\mathbb{R}$ either drift to $\pm\infty$ or oscillate.

The characteristic κ_ξ

Recall: Lévy processes in $\mathbb{R}$ either drift to $\pm\infty$ or oscillate.

MAPs often show a similar behaviour:
Define the long-term mean (here for the MAP $(\xi_t, J_t)_{t\geq 0}$)

$$\kappa_\xi := \sum_{j\in\mathcal{S}} \pi_j \left(\gamma_{\xi^{(j)}} + \int_{|x|\geq 1} x \nu_{\xi^{(j)}}(dx) \right) + \sum_{\substack{(i,j)\in\mathcal{S}\times\mathcal{S} \\ i\neq j}} \pi_i q_{i,j} \int_{\mathbb{R}} x F_\xi^{(i,j)}(dx).$$

Whenever $\mathcal{S}$ is finite, κ_ξ fully determines the long-term behaviour of ξ:

$$\kappa_\xi > 0 \quad \Rightarrow \quad \lim_{t\to\infty} \xi_t = \infty \;\; \mathbb{P}_\pi\text{-a.s.},$$

$$\kappa_\xi < 0 \quad \Rightarrow \quad \lim_{t\to\infty} \xi_t = -\infty \;\; \mathbb{P}_\pi\text{-a.s.}, \quad \text{while}$$

$$\left.\begin{array}{r} \kappa_\xi = 0 \text{ and} \\ \mathbb{P}_\pi\big(\sup_{t\geq 0} |\xi_t| < \infty\big) < 1 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} \limsup_{t\to\infty} \xi_t = \infty \text{ and} \\ \liminf_{t\to\infty} \xi_t = -\infty \end{array}\right\} \; \mathbb{P}_\pi\text{-a.s.}$$

On $\mathfrak{E}^{(1)}$

Proposition (B&Sideris, '19): Assume that $0 < \kappa_\xi < \infty$ and

$$\sup_{j \in \mathcal{S}} \left(|\gamma_{\eta^{(j)}}| + \sigma^2_{\eta^{(j)}} + \int_{(0,1)} x^2 \nu_{\eta^{(j)}}(dx) \right) < \infty. \qquad (5)$$

Then $\mathfrak{E}^{(1)}(t)$ converges $\mathbb{P}_\pi$-a.s. to a finite random variable as $t \to \infty$.

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Note that

- For $\mathcal{S}$ finite, (5) is always fulfilled.

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Then $\mathfrak{E}^{(1)}(t)$ converges $\mathbb{P}_\pi$-a.s. to a finite random variable as $t \to \infty$.

Note that

- For $\mathcal{S}$ finite, (5) is always fulfilled.
- In general (5) is necessary.

A result for $\mathfrak{E}^{(2)}$

Proposition (B&Sideris, '19): Assume $0 < \kappa_\xi < \infty$.

1. $\mathfrak{E}^{(2)}(t)$ converges $\mathbb{P}_\pi$-a.s. to a finite random variable as $t \to \infty$ if and only if for some/all $j \in \mathcal{S}$

$$\int_{(1,\infty)} \log q \, \mathbb{P}_j\Big(\sup_{0<t\leq\tau_1(j)} e^{-\xi_{t-}} |\Delta(Y_t^{b,\eta} + Y_t^{(2)})| \in dq \Big) < \infty.$$

2. $\mathfrak{E}^{(2)}(t)$ converges in $\mathbb{P}_j$-probability to some random variable $\mathfrak{E}_\infty^{(2)}$ for all $j \in \mathcal{S}$, if and only if for some/all $j \in \mathcal{S}$

$$\int_{(1,\infty)} \log q \, \mathbb{P}_j\Big(\Big| \int_{(0,\tau_1(i)]} e^{-\xi_{t-}} d(Y_t^{b,\eta} + Y_t^{(2)}) \Big| \in dq \Big) < \infty.$$

Open questions/Outlook

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- Fluctuation theory for MAPs with countable state space?

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- Fluctuation theory for MAPs with countable state space?
- $\mathfrak{E}$ as stationary distribution of a Markov modulated generalized Ornstein-Uhlenbeck process?

Thank you
for your attention!

Main References

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- **A. Behme and A. Sideris (2019+)**
  Exponential functionals of Markov additive processes (working title).
  To be submitted soon.