Connections between the Dirichlet and the Neumann problem for integrable boundary data

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Abstract: Equivalence between the Dirichlet and Neumann problem

- We present a **representation of the solution of the Neumann problem** for the Laplace operator on the d-dimensional unit ball in terms of the solution of an associated Dirichlet problem.

- The representation is suitable for **extension to smooth planar domains**.

- The representation holds in the case of **integrable boundary data**, to provide an explicit solution of the generalized Neumann problem.

- A **new proof for the Brosamler formula**, the probabilistic representation of the solution of the Neumann problem.

- A **Brosamler-type formula** in terms of the free Brownian motion.

- Explicit **representation of Dirichlet-to-Neumann operator.**

- The representation holds for **other operators** besides the Laplacian.

Preliminaries

For a smooth bounded domain $D \subset \mathbb{R}^d$, $d \geqslant 1$, consider the *Dirichlet and Neumann problems* for the Laplacian in D:

$$\begin{cases} \Delta u = 0 \ \text{ in } \ D \\ u = \varphi \ \text{ on } \ \partial D \end{cases},$$

respectively

$$\begin{cases} \Delta U = 0 \ \text{ in } \ D \\ \frac{\partial U}{\partial \nu} = \Phi \ \text{ on } \ \partial D \end{cases},$$

where ν is the outward unit normal to the boundary of D, and $\varphi, \phi : \partial D \to \mathbb{R}$ are the given boundary values.

$D = \mathbb{U} = \left\{ z \in \mathbb{R}^d : |z| < 1 \right\}$

The Dirichlet problem has a unique solution for continuous boundary data φ on $\partial\mathbb{U}$

The Neumann problem also has a solution (unique up to additive constants), for continuous boundary data Φ on $\partial\mathbb{U}$, satisfying the condition

$$\int_{\partial\mathbb{U}} \Phi(z)\,\sigma(dz) = 0.$$

The centering condition is a necessary condition for the existence of a solution, since by Green's first identity we have

$$\int_{\partial\mathbb{U}} \Phi(z)\,\sigma(dz) = \int_{\partial\mathbb{U}} 1\,\frac{\partial U}{\partial\nu}(z)\,\sigma(dz) = \int_{\mathbb{U}} 1\Delta U(z) + \nabla 1 \cdot \nabla U(z)\,dz = 0.$$

Probabilistic representation of the solution of the Dirichlet problem

$$u(x) = \mathbb{E}^x \varphi(B_{\tau_D}),$$

where $(B_t)_{t \geqslant 0}$ is a d-dimensional Brownian motion starting at $B_0 = x$ and $\tau_D = \inf\{t > 0 : B_t \notin D\}$.

[Shizuo Kakutani, Two-dimensional Brownian motion and harmonic functions. *Proc. Imp. Acad. Tokyo* **20** (1944)]

Probabilistic representation of the solution of the Neumann problem; Brosamler formula

$$U(x) = \lim_{t\to\infty} \frac{1}{2}\mathbb{E}^x \int_0^t \Phi(X_s)dL_s \quad \text{(Brosamler formula)},$$

where $(X_t)_{t\geqslant 0}$ is the reflecting Brownian motion in D starting at $X_0 = x$ and $(L_t)_{t\geqslant 0}$ is the boundary local time of X.

[G. A. Brosamler, A probabilistic solution of the Neumann problem, *Math. Scand.* 1976]

[A. P. Korostelev, *Th. Probab. & Appl.* 1973]

[D. S. Jerison, C. E. Kenig, *Studies in PDEs* 1982], [R. F. Bass, P. Hsu, *Ann. Probab.*, 1991]: extension of Brosamler formula to Lipschitz domains

[Benchérif-Madani, A., Pardoux, E, *Stoch. Analysis & Appl. 2009*]: extension of Brosamler formula for some nonlinear Neumann boundary value problems

Theorem (the case of the unit ball $\mathbb{U} \subset \mathbb{R}^d$)

(i) *Assume that* $\Phi : \partial\mathbb{U} \to \mathbb{R}$ *is continuous and satisfies* $\int_{\partial\mathbb{U}} \Phi(z)\,\sigma_0(dz) = 0$*. If* u *is the solution of the Dirichlet problem with boundary condition* $\varphi = \Phi$ *on* $\partial\mathbb{U}$*, then*

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho}\,d\rho, \qquad z \in \mathbb{U},$$

is the solution to the Neumann problem with $U(0) = 0$*.*

(ii) *Assume* $\varphi : \partial\mathbb{U} \to \mathbb{R}$ *is continuous and let* U *be the solution of the Neumann problem with boundary condition* $\Phi = \varphi - \int_{\partial\mathbb{U}} \varphi(\xi)\,\sigma_0(d\xi)$*. If we define*

$$u(z) = z \cdot \nabla U(z) + \int_{\partial\mathbb{U}} \varphi(\xi)\,\sigma_0(d\xi), \qquad z \in \mathbb{U},$$

then u *is the solution to the Dirichlet problem.*

[L. Beznea, M.N. Pascu, and N.R. Pascu, *Potential Analysis* **44** (2016)]

[M. Aldawsari, T. Savina, arXiv:1901.00981v1, 2019] extension to arcs

The case of a smooth bounded planar domains

Corollary 2. *Let* $D \subset \mathbb{C}$ *be a smooth bounded simply connected domain* ($C^{1,\alpha}$ *boundary with* $0 < \alpha < 1$ *will suffice), and for an arbitrarily fixed* $w_0 \in D$ *let* $f : \mathbb{U} \to D$ *be the conformal map of the unit disk* $\mathbb{U}$ *onto* D *with* $f(0) = w_0$, $\arg f'(0) = 0$, *and let* $g = f^{-1} : D \to \mathbb{U}$ *be its inverse.*

Assume $\Phi : \partial D \to \mathbb{R}$ *is continuous and* $\int_{\partial D} \Phi(w)\, \sigma_0(dw) = 0$.

If u *is the solution of the Dirichlet problem with boundary condition*

$$\varphi(w) = \frac{1}{|g'(w)|}\Phi(w), \qquad w \in \partial D,$$

then

$$U(w) = \int_0^1 \frac{u(f(\rho g(w)))}{\rho} d\rho, \qquad w \in D,$$

is the solution to the Neumann problem with $U(w_0) = 0$.

The case of integrable boundary data

Aim: To extend the result to a correspondence between the solutions of the Dirichlet problem and the Neumann problem for the unit ball in the general case of integrable boundary data.

- We will use A. Cornea's notion of **controlled convergence** [A. Cornea, *C. R. Acad. Sci. Paris, Ser. I Math.* **320** (1995)]

- We provide an explicit solution to the general Neumann problem for the Laplace operator.

Introduction: classical Dirichlet problem

Let D be a bounded open set in $\mathbb{R}^d$, $d \geqslant 1$.

- If the classical Dirichlet problem with boundary data φ has a solution then φ is a continuous function on ∂D.

- **Zaremba's example of an open set having no solution for the classical Dirichlet problem.** Let D be the punctured unit ball, $D := \mathbb{U} \setminus \{0\}$. Then $\partial D = S \cup \{0\}$ and consider $\varphi : \partial D \to \mathbb{R}$, $\varphi \in C(\partial D)$, given by

$$\varphi(y) := \begin{cases} 0, & \text{if } |y| = 1 \\ 1, & \text{if } y = 0 \end{cases}$$

Then the classical Dirichlet problem has no solution for the boundary data φ.

- Although the classical Dirichlet problem has no solution, if we set $h := H_\varphi^{\mathbb{U}}|_D$ then h is a harmonic function on D such that

$$\lim_{D \ni x \to y} h(x) = \varphi(y) \ \text{ for all } \ y \in \partial D \setminus \{0\}.$$

Definition of controlled convergence

Let $D \subset \mathbb{R}^d$ be a bounded open set, $f : \partial D \to \overline{\mathbb{R}}$ and $h, k : D \to \overline{\mathbb{R}}$, $k \geqslant 0$.

The function h **converges to** f **controlled by** k (we write $h \xrightarrow{k} f$) if the following conditions hold:

For any any point $z_0 \in \partial D$ we have

(a) If $\liminf_{D \ni z \to z_0} k(z) < +\infty$, then $f(z_0) \in \mathbb{R}$ and

$\lim_{D \ni z \to z_0} \frac{h(z) - f(z_0)}{1 + k(z)} = 0$.

(b) If $\lim_{D \ni z \to z_0} k(z) = +\infty$, then $\lim_{D \ni z \to z_0} \frac{h(z)}{1 + k(z)} = 0$.

The function k will be called a *control function* for f.

Remark. A harmonic function h on D is the solution for the classical Dirichlet problem with boundary data f if and only if h converges to f controlled by a bounded function k.

Generalized solution of the Dirichlet problem

A **generalized solution of the Dirichlet problem** is a harmonic function $u : D \to \mathbb{R}$ which converges to φ, controlled by a non-negative harmonic function $k : D \to \mathbb{R}_+$.

The case of the Euclidean ball

• It can be shown that in the case of the unit ball Cornea's approach is equivalent to the Perron-Wiener-Brelot approach for the generalized solution of the Dirichlet problem. More precisely, for integrable boundary data, both methods indicate that:
the generalized solution of the Dirichlet problem is given by the stochastic solution; cf.

[L. Beznea, A. Cornea, M. Röckner, *J. Funct. Anal.* **261** (2011)]

[L. Beznea, The stochastic solution of the Dirichlet problem and controlled convergence. *L. N. of Sem. Interdis. di Mat.* **10** (2011)]

Generalized solution of the Neumann problem

Let $D \subset \mathbb{R}^d$ be a bounded open set, $h, k : D \to \overline{\mathbb{R}}$, $k \geqslant 0$.
We say that the function h **has a continuous extension to** $\overline{D}$ controlled by k if there exists a function $f : \partial D \to \overline{\mathbb{R}}$ such that $h \stackrel{k}{\to} f$.

Definition

A **generalized solution of the Neumann problem** (with controlled convergence) is a harmonic function $U : D \to \mathbb{R}$ which has a continuous extension to ∂D, controlled by a non-negative harmonic function $k : D \to \mathbb{R}_+$, and for any $z_0 \in \partial D$ for which $\limsup_{[0,z_0]\ni z \to z_0} k(z) < +\infty$ we have

$$\lim_{\varepsilon \searrow 0} \frac{U(z_0) - U(z_0 - \varepsilon \nu(z_0))}{\varepsilon} = \Phi(z_0),$$

where $\nu(z)$ denotes the outward unit normal to the boundary of D at $z \in \partial D$.

Theorem

Assume $\Phi : \partial\mathbb{U} \to \mathbb{R}$ *is integrable and satisfies* $\int_{\partial\mathbb{U}} \Phi(z)\,\sigma_0(dz) = 0$.

If u *is the generalized solution of the Dirichlet problem with boundary condition* $\varphi = \Phi$ *on* $\partial\mathbb{U}$, *then*

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \overline{\mathbb{U}},$$

is a generalized solution to the Neumann problem with $U(0) = 0$.

[L. Beznea, M.N. Pascu, and N.R. Pascu, In: *Stochastic Analysis and Related Topics* (Progress in Probability **72**), Springer 2017]

Weak normal derivative

Definition

For a function $u \in C\left(\overline{D}\right) \cap C^2(D)$ with Δu bounded, the function $u^* \in L^\infty(\partial D)$ is called the **weak normal derivative** of u if

$$\int_D \varphi(x)\,\Delta u(x) - u(x)\,\Delta\varphi(x)\,dx = \int_{\partial D} \varphi(x)\,u^*(x)\,\sigma(dx),$$

for any $\varphi \in C^2\left(\overline{D}\right)$ with $\frac{\partial\varphi}{\partial n} = 0$ on ∂D.

Green's second identity shows that an ordinary normal derivative is a weak normal derivative, and there is at most one weak normal derivative up to a.e. equality.

Brosamler formula

G.A. Brosamler (1976) gave a probabilistic representation of the solution of the Neumann problem as follows.

Theorem

Let X_t denotes the reflecting Brownian motion in D and L_t be its boundary local time. The solution of the Neumann problem with weak normal derivative $\Phi \in L^\infty(\partial D)$ such that $\int_{\partial D} \Phi d\sigma_0 = 0$, is given by

$$U(z) = \frac{1}{2} \lim_{t \to \infty} \mathbb{E}^z \int_0^t \Phi(X_s)\, dL_s, \quad z \in D.$$

Preliminaries on the reflecting Brownian motion

Consider a bounded $C^{1,\alpha}$ $(0 < \alpha < 1)$ domain $D \subset \mathbb{R}^d$ and the stochastic differential equation:

$$X_t = X_0 + B_t - \frac{1}{2}\int_0^t \nu(X_s)\, dL_s, \qquad t \geqslant 0, \tag{1}$$

the **Skorohod representation of the reflecting Brownian motion.**

Pathwise uniqueness and the existence of a strong solution of the above SDE for smooth domains is known, results of Stroock–Varadhan, Lions–Sznitman, Bass–Hsu.

More precisely, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider a d-dimensional (free) Brownian motion $(B_t)_{t\geqslant 0}$ starting at the origin, and let $\mathcal{F}^B = \left(\mathcal{F}_t^B\right)_{t\geqslant 0}$ the natural filtration generated by B satisfying the usual hypotheses.

Reflecting Brownian motion in D **starting at** $z \in \overline{D}$ is an $\mathcal{F}^B$-adapted process X satisfying the stochastic differential equation (1), with $X_0 = z$ and $X_t \in \overline{D}$ for all $t \geqslant 0$ almost surely, and $L = L^X$ is the local time of X on the boundary of D (the continuous nondecreasing $\mathcal{F}^B$-adapted process with $L_0 = 0$ which increases only when $X_t \in \partial D$).

New proof of Brosamler formula,

using the representation of the solution of the Neumann problem in terms of the solution of the Dirichlet problem, based on path trajectories of processes, in the case of:

- the unit ball in $\mathbb{R}^d$;

- the smooth bounded simply connected planar domains ($C^{1,\alpha}$ boundary, with $0 < \alpha < 1$).

[L. Beznea, M.N. Pascu, and N.R. Pascu, *Analysis and Mathematical Physics* (2019), to appear]

A main tool in the proof is an explicit description of the boundary local time of reflecting Brownian motion,
[M.N. Pascu, *Trans. Amer. Math. Soc.* (2002)]

Complement to Brosamler formula: Probabilistic representation of the generalized solution of the Neumann problem, using the Brownian motion

Corollary

Assume that $\Phi : \partial\mathbb{U} \to \mathbb{R}$ *is integrable and* $\int_{\partial\mathbb{U}} \Phi(z)\,\sigma_0(dz) = 0$.

Then

$$U(z) = \int_0^1 \frac{\mathbb{E}^{\rho z}\Phi(B_{\tau_D})}{\rho} d\rho, \qquad z \in \overline{\mathbb{U}},$$

is a generalized solution to the Neumann problem (with controlled convergence) with $U(0) = 0$.

Extension to a second-order partial differential operator

$$\mathcal{L}f(z) = \sum_{i,j=1}^{d} a_{ij}(z) \frac{\partial^2 f}{\partial z_i \partial z_j}(z) + \sum_{i=1}^{d} a_i(z) \frac{\partial f}{\partial z_i}(z),$$

where the coefficients a_{ij} are smooth and homogeneous of degree $k \in [0, 1]$, i.e.

$$a_{ij}(\rho z) = \rho^k a_{ij}(z), \qquad 0 \le \rho \le 1, z \in \mathbb{U}, 1 \le i, j \le d,$$

and the coefficients a_i are also smooth and homogeneous of degree $k - 1$, i.e.

$$a_i(\rho z) = \rho^{k-1} a_i(z), \qquad 0 \le \rho \le 1, z \in \mathbb{U}, 1 \le i \le d.$$

If u and U are related by $U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho$, $z \in \mathbb{U}$, it can be checked that

$$\mathcal{L}U(z) = \int_0^1 \rho^{1-k} \mathcal{L}u(\rho z)\, d\rho \quad \text{and} \quad \frac{\partial U}{\partial \nu}(z) = u(z), \qquad z \in \mathbb{U}.$$

Theorem

Assume $\Phi : \partial\mathbb{U} \to \mathbb{R}$ *is continuous. If* u *is the solution of the Dirichlet problem*

$$\begin{cases} \mathcal{L}u = 0 \text{ in } D \\ u = \Phi \text{ on } \partial D \end{cases}$$

and $u(0) = 0$ *then*

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \mathbb{U},$$

is the solution to the Neumann problem

$$\begin{cases} \mathcal{L}U = 0 \text{ in } D \\ \frac{\partial U}{\partial \nu} = \Phi \text{ on } \partial D \end{cases},$$

with $U(0) = 0$.