Recurrence of critical multi-dimensional affine processes

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Affine recursion

The **Affine process** is the Markov chain in $\mathbb{R}^d$

$$X_n^x = A_n X_{n-1}^x + B_n \in \mathbb{R}^d \qquad X_0^x = x$$

where $(A_n, B_n) \in M(d \times d) \times \mathbb{R}^d$ are an i.i.d sequence.

Question : Find conditions that ensure that X_n^x is **recurrent** i.e.

$$\exists\ K > 0 \text{ such that } \mathbb{P}(|X_n^x| < K \text{ infinetely often }) = 1$$

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- $X_n^x = A_n B_{n-1} + A_n A_{n-1} B_{n-2} + \cdots + A_n \cdots A_1 x$

Lyapunov exponent $\lambda_A = \lim_{n \to \infty} \frac{1}{n} \log \|A_n \cdots A_1\|$ a.s.

- **Contracting case :** $\lambda_A < 0$ i.e. $\|A_n \cdots A_1\| \sim e^{\lambda n} \to 0$
  $\hookrightarrow$ Recurrence

Recurrence in the critical case

Critical case : $\lambda_A = 0$
$\underline{\lim}\|A_n \cdots A_1\| = 0$ and $\overline{\lim}\|A_n \cdots A_1\| = +\infty$

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Under suitable irreducibility and moment hypothesis ($\mathbb{E}(\|A_1\|^\delta) < \infty$...) and in the critical case $\lambda_A = \lim_{n\to\infty} \frac{1}{n} \log \|A_n \cdots A_1\| = 0$,
the Markov chain $X_n^x = A_n X_{n-1}^x + B_n$ is recurrent in the following cases :

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- the maps $x \mapsto A_n x + B_n$ are <u>similarities</u> (i.e $A_n \in \mathbb{R}^* \times O(d)$)

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- the matrices A_n have positive coefficient and "strongly contract" the cone $\mathbb{R}_+^d$
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- (the matrices A_n have rank 1)

Corollary. There exists an invariant Radon measure

Contracting case

Recurrence in the Contracting case (Kesten '73)

If $\mathbb{E}(\log^+ \|A_n\| + \log^+ |B_n|) < \infty$ and $\lambda_A = \lim_{n\to\infty} \frac{1}{n} \log \|A_n \cdots A_1\| < 0$ then X_n^x is recurrent.

Idea of the Proof :

Recurrence criterium

X_n^x is recurrent if

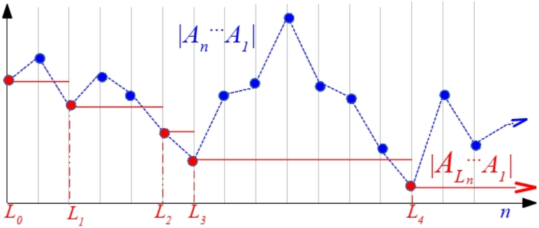
1. $|X_n^x - X_n^y| \to 0$ a.s., uniformly on x, y in compact set
2. $|X_n^0| \leq_{st} Z$ for some finite RV Z

1. $|X_n^x - X_n^y| \leq \|A_n \cdots A_1\| |x - y| \sim \mathrm{e}^{\lambda n} |x - y| \to 0$
2. $$\begin{aligned} |X_n^0| &= |A_n B_{n-1} + A_n A_{n-1} B_{n-2} + \cdots + A_n \cdots A_2 B_1| \\ &=_{st} |A_1 B_2 + A_1 A_2 B_3 + \cdots + A_1 \cdots A_{n-1} B_n| \\ &\leq \|A_1\| |B_2| + \cdots + \|A_1 \cdots A_{n-1}\| |B_n| + \cdots =: Z < \infty \end{aligned}$$

Critical case in one dimension (Grincevicius, Elie '82)

Ladder times for $d = 1$

$$L_1 := \inf\{k > 0 \,|\, |A_k \cdots A_1| < 1\}$$
$$L_n := \inf\{k > L_{n-1} \,|\, |A_k \cdots A_1| < |A_{L_{n-1}} \cdots A_1|\}$$



Suppose $\mathbb{E}\left(\log^+|A_n| + \log^+|B_n|\right)^{2+\varepsilon} < \infty$ then

1. $|X^x_{L_n} - X^y_{L_n}| = |A_{L_n} \cdots A_1||x-y| \to 0$
2. $|X^0_{L_n}| \leq_{st} Z < \infty$

$\Rightarrow X^x_{L_n}$ recurrent $\Rightarrow X^x_n$ recurrent

Remark : 2 needs a delicate moment control and uses $\mathbb{E}(L_1^{\frac{1}{2+\varepsilon}}) < \infty$

Ladder times for $d > 1$

Ladder times in higher dimension

Let $\xi_0 \in \mathbb{P}^{d-1}(\mathbb{R})$

$$L_1(\xi_0) := \inf\{k > 0 \,|\, |A_k \cdots A_1 \xi_0| < 1\}$$
$$L_n(\xi_0) := \inf\{k > L_{n-1} \,|\, |A_k \cdots A_1 \xi_0| < |A_{L_{n-1}} \cdots A_1 \xi_0|\}$$

- $\mathbb{E}(L_1(\xi_0)^{\frac{1}{2+\varepsilon}}) < \infty$ (Grama, Le Page, Peigné '16 ; Pham '17)
- New problems we need to deal with
  - **Dependence on the direction** $\xi_n = A_n \cdot \xi_{n-1} \in \mathbb{P}^{d-1}$:
    - ★ $X^x_{L_n}$ is not a Markov Chain, but $(X^x_{L_n}, \xi_{L_n}) \in \mathbb{R}^d \times \mathbb{P}^{d-1}(\mathbb{R})$ is
    - ★ $L_n - L_{n-1}$ are not i.i.d
  - **Control of the norm** :

$$|A_n \cdots A_1 \xi_0| \leq \|A_n \cdots A_1\| \text{ but how } \|A_n \cdots A_1\| \overset{?}{\lesssim} |A_n \cdots A_1 \xi_0|$$

Inverse norm control coefficient

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$$C_\infty(\xi_0) := \sup_{n\in\mathbb{N}} \frac{\|A_n \cdots A_1\|}{|A_n \cdots A_1 \xi_0|}$$

$$|A_n \cdots A_1 \xi_0| \le \|A_n \cdots A_1\| \le C_\infty(\xi_0) |A_n \cdots A_1 \xi_0|$$

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Fundamental proposition

Suppose that $A_1\cdot\xi_0 \overset{Law}{=} \xi_0$ and

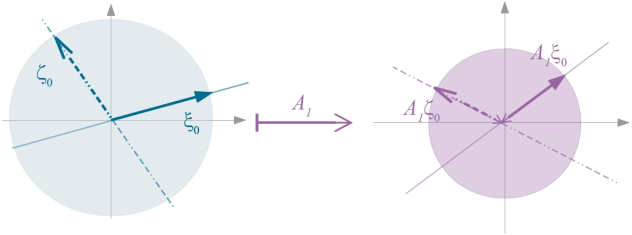
$$\mathbb{E}(L_1(\xi_0)^{\frac{1}{2+\varepsilon}}) < \infty \quad \mathbb{E}(\log^+|B_n|^{3+\delta_\epsilon}) < \infty \quad \mathbb{E}(\log^+|C_\infty(\xi_0)|^{3+\delta_\epsilon}) < \infty$$

Then $X^x_{L_n}$ and X^x_n are recurrent.

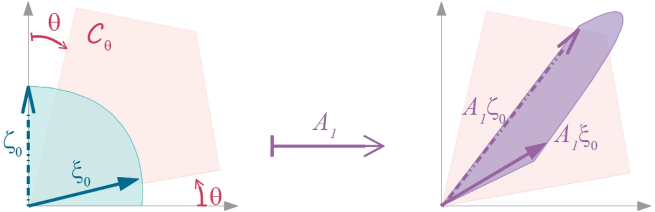
Norm control on "nice" semi-groups of matrices

$$C_\infty(\xi_0) := \sup_{n\in\mathbb{N}} \frac{\|A_n \cdots A_1\|}{|A_n \cdots A_1 \xi_0|}$$

- Similarities $\Rightarrow C_\infty(\xi_0) \equiv 1$
  In fact if $A_n = a_n R_n$ with $a_n \in \mathbb{R}_+^*$ and $R_n \in O(d)$ then
  $\frac{|A_n\cdots A_1\zeta_0|}{|A_n\cdots A_1\xi_0|} = \frac{a_n\cdots a_1}{a_n\cdots a_1} = 1$



- Positive matrices such that $A_n(\mathbb{R}_+^d) \subset \mathcal{C}_\theta \Rightarrow C_\infty(\xi_0) \leq Cst_\theta \ \forall \xi_0 \in \mathcal{C}_\theta$



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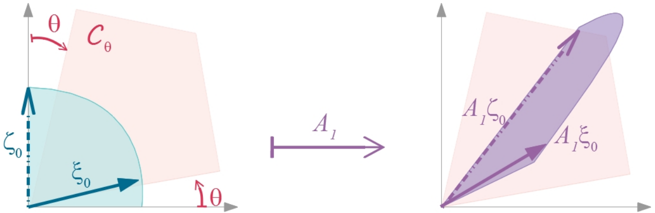
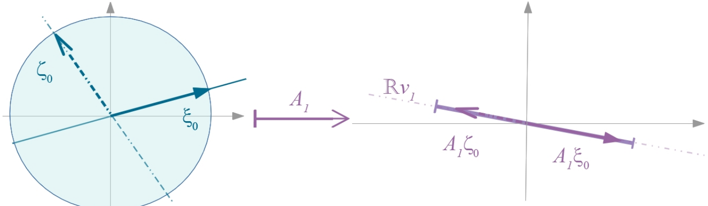
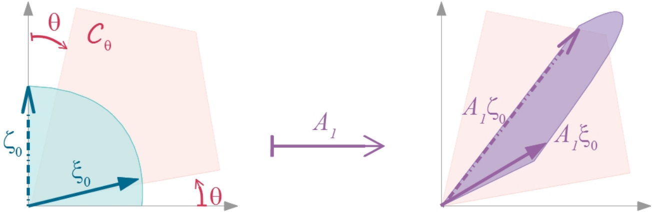


- Rank 1 matrices. If $A_n x = a_n \langle \mathbf{v}_n, x \rangle \mathbf{v}_n$

$$\frac{|A_1 \zeta_0|}{|A_1 \xi_\mathbf{0}|} = \frac{a_1 |\langle \mathbf{v}_1, \zeta_0 \rangle|}{a_1 |\langle \mathbf{v}_1, \xi_\mathbf{0} \rangle|} = \frac{|\langle \mathbf{v}_1, \zeta_0 \rangle|}{|\langle \mathbf{v}_1, \xi_\mathbf{0} \rangle|} \leq \frac{|\zeta_0|}{|\langle \mathbf{v}_1, \xi_\mathbf{0} \rangle|}$$

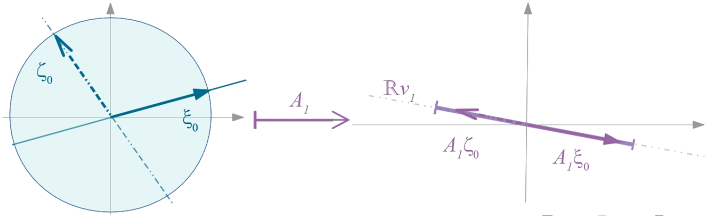


- Positive matrices such that $A_n(\mathbb{R}^d_+) \subset \mathcal{C}_\theta \Rightarrow C_\infty(\xi_0) \leq Cst_\theta \ \forall \xi_0 \in \mathcal{C}_\theta$



- Rank 1 matrices. If $A_n x = a_n \langle \mathbf{v}_n, x\rangle \mathbf{v}_n \Rightarrow C_\infty(\xi_0) = \frac{1}{|\langle \mathbf{v}_1, \xi_0\rangle|}$

$$\frac{|A_n \cdots A_1 \zeta_0|}{|A_n \cdots A_1 \xi_\mathbf{0}|} = \frac{a_n \cdots a_1 |\langle \mathbf{v}_n, \mathbf{v}_{n-1}\rangle| \cdots |\langle \mathbf{v}_1, \zeta_0\rangle|}{a_n \cdots a_1 |\langle \mathbf{v}_n, \mathbf{v}_{n-1}\rangle| \cdots |\langle \mathbf{v}_1, \xi_\mathbf{0}\rangle|} = \frac{|\langle \mathbf{v}_1, \zeta_0\rangle|}{|\langle \mathbf{v}_1, \xi_\mathbf{0}\rangle|} \leq \frac{|\zeta_0|}{|\langle \mathbf{v}_1, \xi_\mathbf{0}\rangle|}$$

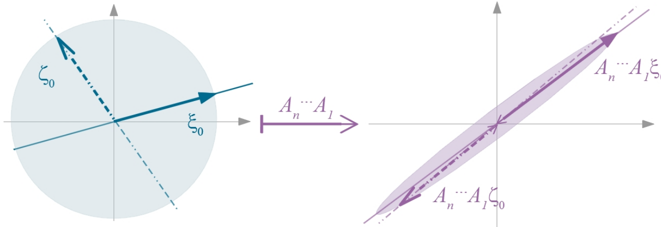


Norm control coefficient in $GL_d(\mathbb{R})$

Let $A_n \in GL(d, \mathbb{R})$ and suppose $\mathbb{E}(\|A_n\|^\delta) + \mathbb{E}(\|A_n^{-1}\|^\delta) < \infty$ for some $\delta > 0$ + suitable irreducibility conditions then for all $p > 1$
$\mathbb{E}(|\log C_\infty(\xi_0)|^p) < \infty$

Idea of the proof : $S_n := \log \frac{|A_n \cdots A_1 \zeta_0|}{|A_n \cdots A_1 \xi_0|} = \log \frac{|A_n \zeta_{n-1}|}{|A_n \xi_{n-1}|} + \cdots + \log \frac{|A_1 \zeta_0|}{|A_1 \xi_0|}$.
Using strong contraction of A_n on $\mathbb{P}^{d-1} \Rightarrow \exists\ \rho < 1$:

$$\mathbb{E}(\sup_n |S_n|^p)^{1/p} \leq \sum_{n=0}^{\infty} \mathbb{E}\left(\left|\log \frac{|A_n \zeta_{n-1}|}{|A_n \xi_{n-1}|}\right|^p\right)^{1/p} \leq Cst \sum_{n=0}^{\infty} \rho^n < \infty$$



Summarizing :

- If one can control the moment of

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  one can prove that $X_n = A_n X_{n-1} + B_n$ is recurrent.
- With moment of type $\mathbb{E}(\|A_n\|^\delta) < \infty$ + irreducibility, this can be done in several different situations : if the A_n are positive, or are invertible, etc.

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On going works and open questions :

- Unify the proofs, weaken moment hypothesis.
Conjecture : Recurrence should hold supposing only log-moments.
- Local contraction property

$$\lim_{n \to \infty} |X_n^x - X_n^y| 1_{|X_n^x| < K} = 0 \quad \text{a.s.}$$