Chernoff approximation of diffusions and further applications

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Survey paper

Yana A. Butko (2019)
The method of Chernoff approximation
ArXiv: http://arxiv.org/abs/1905.07309.

Chernoff approximation of Markov evolution

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- $(\xi_t)_{t \geqslant 0}$ is a time homogeneous Markov process $\quad (\Rightarrow$ no memory)
  $\Rightarrow$ transition kernel $\; P(t, x, dy) := \mathbb{P}(\xi_t \in dy \mid \xi_0 = x)$

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  $\Rightarrow$ transition kernel $\;P(t,x,dy) := \mathbb{P}(\xi_t \in dy \mid \xi_0 = x)$
- Then
  $$f(t,x) := \int f_0(y)P(t,x,dy) \equiv \mathbb{E}\left[f_0(\xi_t) \mid \xi_0 = x\right]$$
  solves the following evolution equation:
  $$\begin{cases} \frac{\partial f}{\partial t}(t,x) & = & Lf(t,x), \\ f(0,x) & = & f_0(x), \end{cases}$$
  where
  $$T_t f_0(x) := \int f_0(y)P(t,x,dy) \equiv e^{tL} f_0.$$
  $(T_t)_{t\geqslant 0}$ is an operator semigroup (i.e. $T_0 = \mathrm{Id}$, $T_t \circ T_s = T_{t+s}$).

Chernoff approximation of Markov evolution

Stochastics

To determine the transition kernel $P(t, x, dy)$ for a given process $(\xi_t)_{t\geqslant 0}$.

$\Updownarrow$

Functional Analysis

To construct the semigroup $T_t \equiv e^{tL}$ with a given generator L.

$\Updownarrow$

PDEs

To solve a (Cauchy problem for a) given PDE $\frac{\partial f}{\partial t} = Lf$.

Chernoff approximation of Markov evolution

Example:

- Heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f, \qquad x \in \mathbb{R}^d.$$

- Heat semigroup

$$T_t f_0(x) := (2\pi t)^{-d/2} \int\limits_{\mathbb{R}^d} f_0(y) \exp\left\{-\frac{|x-y|^2}{2t}\right\} dy.$$

- Transition kernel of Brownian motion

$$P(t, x, dy) = (2\pi t)^{-d/2} \exp\left\{-\frac{|x-y|^2}{2t}\right\} dy.$$

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Chernoff approximation: To find $(F(t))_{t\geqslant 0}$ (not a SG!!!) such that

$$T_t f_0 = \lim_{n\to\infty} \left[F(t/n)\right]^n f_0.$$

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$\Rightarrow$ discrete time approximation to the solution $f(t,x)$:

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$\Rightarrow$ Markov chain approximation to $(\xi_t)_{t\geqslant 0}$ (e.g., Euler scheme),

$$(\xi_k^n)_{k=1,\dots,n}, : \qquad \mathbb{E}\left[f_0(\xi_k^n)\,|\,\xi_{k-1}^n\right] = F(t/n)f_0(\xi_{k-1}^n)$$

$$\Rightarrow \qquad \mathbb{E}\left[f_0(\xi_t)\,|\,\xi_0 = x\right] = \lim_{n\to\infty} \mathbb{E}\left[f_0(\xi_n^n)\,|\,\xi_0 = x\right]$$

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$\Rightarrow$ approximation of path integrals in Feynman-Kac formulae.

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Chernoff Theorem (1968): Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semigroup on X with generator $(L, \mathrm{Dom}(L))$. Let $(F(t))_{t\geqslant 0}$ be a family of bounded linear operators on X. Assume that

- $F(0) = \mathrm{Id}$,
- $\|F(t)\| \leqslant e^{wt}$ for some $w \in \mathbb{R}$, and all $t \geqslant 0$,
- $\lim\limits_{t\to 0} \dfrac{F(t)\varphi - \varphi}{t} = L\varphi$

  for all $\varphi \in D$, where D is a core for $(L, \mathrm{Dom}(L))$.

Then it holds

$$T_t\varphi = \lim_{n\to\infty} [F(t/n)]^n \varphi, \quad \forall\ \varphi \in X,$$

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- $F(0) = \mathrm{Id}$, *(consistency)*
- $\|F(t)\| \leqslant e^{wt}$ for some $w \in \mathbb{R}$, and all $t \geqslant 0$, *(stability)*
- $\lim\limits_{t\to 0} \dfrac{F(t)\varphi - \varphi}{t} = L\varphi$ *(consistency)*

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Meta-theorem of Numerics: Consistency + stability $\Rightarrow$ convergence.

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$$L \text{ is bdd} \quad \Rightarrow \quad F(t) := \mathrm{Id} + tL \quad \sim \quad e^{tL} \quad \Rightarrow$$

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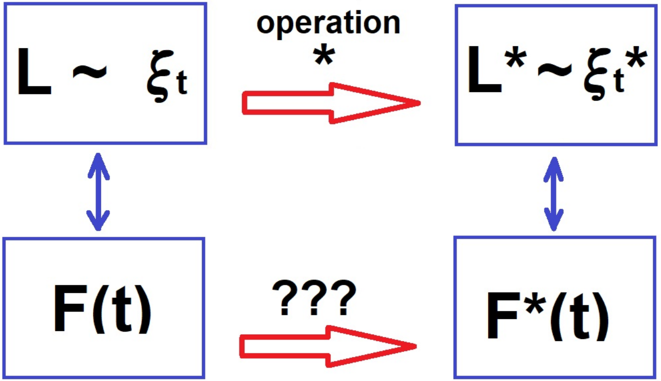

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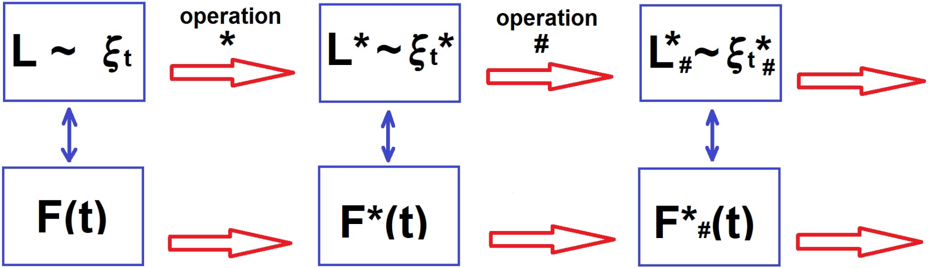

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   - Feller processes in $\mathbb{R}^d$ (Böttcher, Butko, Schilling, Schnurr, Smolyanov 2009-2012)
   - Brownian motion in a compact Riemannian manifold (Smolyanov, Weizsäcker, Wittich 1999-2007)

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- operator splitting: $L^* := L_1 + \ldots + L_m$
- averaging of generators: $L^* := \int L_\varepsilon \mu(d\varepsilon)$
- multiplicative perturbations of L $\Leftrightarrow$ random time change of ξ_t via an additive functional: $L^* := aL$
- subordination: $L^* := -f(-L)$, $\xi^*_t := \xi_{\eta_t}$
- "rotation": $L^* := iL$
- killing of ξ_t upon leaving a domain $G \subset \mathbb{R}^d$:
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  Borisov, Orlov, Sakbaev, Smolyanov 2014-2018
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If $F_k(t)$: $\quad F_k(0) = \mathrm{Id}, \quad \|F_k(t)\| \leqslant e^{tw_k}, \quad F_k'(0) = L_k \quad$ on D

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Corollary (Daletskii–Lie–Trotter formula):

$$e^{tL_1} \circ e^{tL_2} \sim e^{t(L_1+L_2)} \quad \text{i.e.} \quad e^{t(L_1+L_2)} = \lim_{n\to\infty} \left[e^{tL_1/n} \circ e^{tL_2/n} \right]^n$$

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Let $m = 2$. Then $F^*(t) \sim e^{tL^*}$ for

$$F^*(t) := \tau F_1(t) \circ F_2(t) + (1-\tau) F_2(t) \circ F_1(t), \quad \tau \in [0,1]$$

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$\theta = \frac{1}{2} \quad \Rightarrow \quad$ symmetric Strang splitting (scheme of 2nd order)

Averaging

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$\mathcal{E}$ is a parameter set, μ is a probability measure on $\mathcal{E}$

$$L^* := \int\limits_{\mathcal{E}} L_\varepsilon \mu(d\varepsilon)$$

Then

$$F(t) := \int\limits_{\mathcal{E}} e^{tL_\varepsilon} \mu(d\varepsilon) \quad \sim \quad e^{t \int_{\mathcal{E}} L_\varepsilon \mu(d\varepsilon)} \equiv e^{tL^*}$$

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Feller process $\xi_t \quad \leftrightsquigarrow T_t \equiv e^{tL}$ on $C_\infty(\mathbb{R}^d)$ with $L \equiv -\widehat{H}$:

$$\widehat{H}\varphi(x) := (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{ip\cdot(x-q)} H(x,p)\varphi(q)\,dq\,dp,$$

$$\equiv \left(\mathcal{F}^{-1} \circ H(x,\cdot) \circ \mathcal{F}\varphi\right)(x)$$

where $H(x,\cdot)$ is given by the Lévy–Khintchine formula

$$H(x,p) = C(x) + iB(x)\cdot p + p\cdot A(x)p + \int\limits_{y\neq 0} \left(1 - e^{iy\cdot p} + \frac{iy\cdot p}{1+|y|^2}\right) N(x,dy).$$

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Theorem:

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Remark: Let $\mu_t^x : \mathcal{F}[\mu_t^x](p) = e^{-tH(x,-p)-ip\cdot x}$. Then

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Example: non-degenerate diffusion $(N \equiv 0,\ C \equiv 0)$:

$$F(t)\varphi(x) = \frac{1}{\sqrt{(4\pi t)^d \det A(x)}}\int\limits_{\mathbb{R}^d}\varphi(q)e^{-\frac{(x-q-tB(x))\cdot A^{-1}(x)(x-q-tB(x))}{4t}}dq$$

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Euler–Maruyama scheme on $[0, t]$ with time step t/n:

$$X_0 := \xi_0, \qquad X_{k+1} := X_k - B(X_k)\frac{t}{n} + \sqrt{\frac{2t}{n}A(X_k)}Z_k, \quad k = 0, \dots, n-1,$$

$(Z_k)_{k=0,\dots,n-1}$ are i.i.d., $\sim N(0, \mathrm{id})$, $X_k \perp Z_k$.

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$$X_0 := \xi_0, \qquad X_{k+1} := X_k - B(X_k)\frac{t}{n} + \sqrt{\frac{2t}{n}A(X_k)}Z_k, \quad k = 0,\dots,n-1.$$

Then, for all $k = 0,\dots,n-1$ holds:

$$\mathbb{E}[f_0(X_{k+1}) \,|\, X_k] = \mathbb{E}\left[f_0\left(x - B(x)\frac{t}{n} + \sqrt{\frac{2t}{n}A(x)}Z_k\right)\right]\Bigg|_{x:=X_k} = F(t/n)f_0(X_k)$$

Chernoff approximation for Feller processes

Non-degenerate diffusion: $F(t) \sim e^{tL}, \qquad L := \mathrm{tr}(A\nabla^2) - B\cdot\nabla$

$$F(t)\varphi(x) = \frac{1}{\sqrt{(4\pi t)^d \det A(x)}} \int_{\mathbb{R}^d} \varphi(q) e^{-\frac{(x-q-tB(x))\cdot A^{-1}(x)(x-q-tB(x))}{4t}} dq$$

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Hence (tower property)

$$\mathbb{E}[f_0(X_n)\,|\,X_0 = x] = \mathbb{E}[\mathbb{E}[f_0(X_n)\,|\,X_{n-1}]\,|\,X_0 = x] = \dots = F^n(t/n)f_0(x).$$

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And

$$\mathbb{E}[f_0(\xi_t)\,|\,\xi_0 = x] = e^{tL}f_0(x) = \lim_{n\to\infty} F^n(t/n)f_0(x) = \lim_{n\to\infty} \mathbb{E}[f_0(X_n)\,|\,X_0 = x].$$

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Chernoff approximation for diffusions with BC

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Task: $\quad$ to find $\quad F^*(t) \sim e^{tL^*} \quad$ on Y

Solution:

1. Find an extension $E^* \; : \; Y \to X$ such that E^* is a linear contraction and there is a core D^* of L^* with

$$E^*(D^*) \subset D$$

Chernoff approximation for diffusions with BC

ξ_t process in $\mathbb{R}^d$ $\Rightarrow$ e^{tL} on $X := C_\infty(\mathbb{R}^d)$

G is a nice domain in $\mathbb{R}^d$

$\xi_t^* :=$ process ξ_t in G + BC $\Rightarrow$ e^{tL^*} on $Y := C(\overline{G})$ or $C_0(G)$

We have: $F(t) \sim e^{tL}$ on X ($\Rightarrow$ $F'(0) = L$ on D)

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Solution:

1. Find an extension $E^* : Y \to X$ such that E^* is a linear contraction and there is a core D^* of L^* with
$$E^*(D^*) \subset D$$
2. We have
$$F^*(t) := R_t \circ F(t) \circ E^* \quad \sim \quad e^{tL^*}$$

$R_t : X \to Y$ is a restriction to $\overline{G}$ / multiplication with $\psi_t \to 1_G$

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We have: $\quad F(t) \sim e^{tL} \quad$ on $X \qquad (\Rightarrow \quad F'(0) = L$ on $D)$

Task: to find $\quad F^*(t) \sim e^{tL^*} \quad$ on Y

Solution:

1. Find an extension $E^* : Y \to X$ such that E^* is a linear contraction and there is a core D^* of L^* with
$$E^*(D^*) \subset D$$
E^* are known for:
   - Dirichlet BC $\quad \varphi = 0 \quad$ on ∂G
   - Robin BC $\quad \frac{\partial \varphi}{\partial n} + \beta\varphi = 0 \quad$ on ∂G, $\quad \beta \geqslant 0$ smooth
2. We have
$$F^*(t) := R_t \circ F(t) \circ E^* \quad \sim \quad e^{tL^*}$$

$R_t : X \to Y$ is a restriction to $\overline{G}$ / multiplication with $\psi_t \to 1_G$

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