# Oil and water on vertex-transitive graphs

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# Model

Two-type internal aggregation model:

**OIL** particles and **WATER** particles, distinguishable.

Pick:

- A graph $G$;
- A parameter $\mu > 0$.

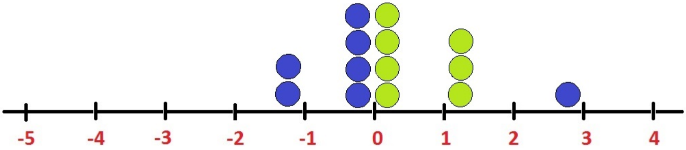
For each $x \in V(G)$ place:

- A random number of OILS: Random variable $\sim \text{Poi}(\mu/2)$, independently of everything else;
- A random number of WATERS: Random variable $\sim \text{Poi}(\mu/2)$, independently of everything else.

# Dynamics

Each vertex $x$ containing *at least an oil* *and* *at least a water* fires the 2 particles, i.e., sends the two particles *independently of each other* to one of the neighbors, chosen uniformly at random.
Any vertex which is either empty, or contains only particles of the same type, does not fire.
Continue inductively as long as there are firings.
**Note: the model is Abelian (order of firings does not change the final configuration)**

# Example on $\mathbb{Z}$



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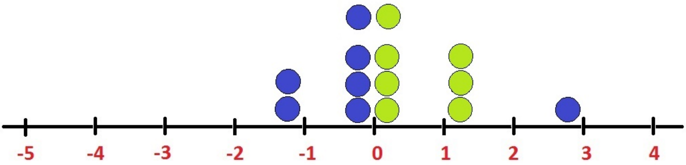

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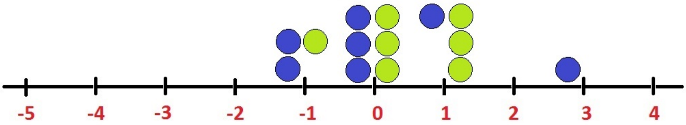

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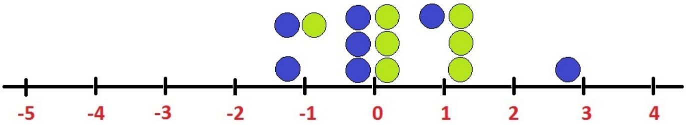

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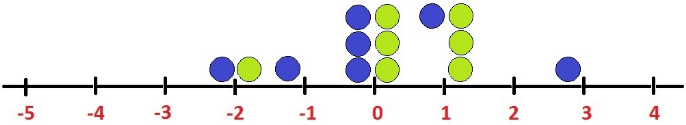

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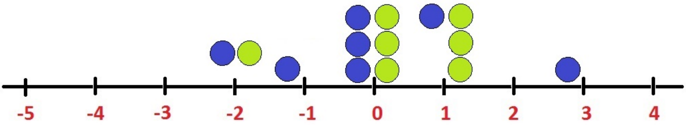

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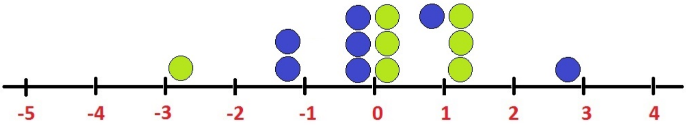

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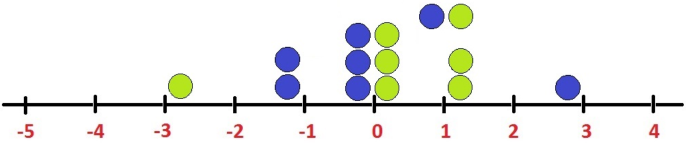

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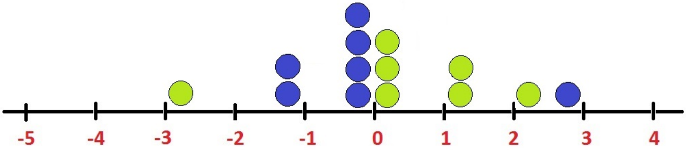

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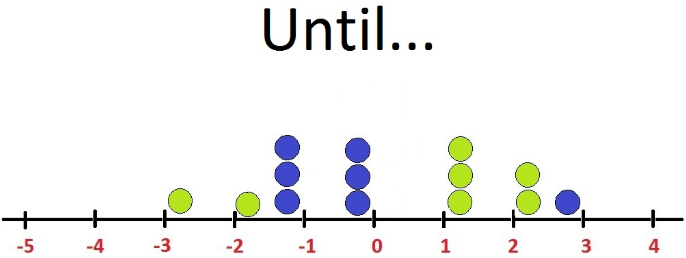

# Question

What is the long-time behavior of the model?

Regime for **Fixation**? Regime for **Activity**?

Where:

- Fixation: any vertex fires only a finite number of times during the whole process;
- Activity: any vertex fires infinitely often.

# Comments

1. Oil-water model introduced by Bond and Levine in the framework of *Abelian networks*.
2. Candellero-Ganguly-Hoffman-Levine: analysis of several statistics of oil-water model on $\mathbb{Z}$, with finite initial configuration consisting of $N$ oil-water couples at the origin.
3. Connections with other interacting particle systems such as *Activated Random Walks, Abelian Sandpiles*... $\Rightarrow$ natural to conjecture phase transition.

# Natural guess

- Intuitively, if $\mu$ is small, there are "few" particles in the system, thus they won't interact very much,

$$\text{We expect } \mu \text{ small } \Rightarrow \text{ Fixation.}$$

- If $\mu$ is large, there are "a lot" of particles in the system, thus they will interact very much,

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**THIS GUESS IS WRONG!**

# Result

**Theorem [CST 19+]** If $G$ is an infinite, vertex-transitive, locally finite graph, then

$$\text{for all } \mu > 0, \quad \mathbb{P}\left[\text{oil-water model fixates}\right] = 1.$$

# Result

**Theorem [CST 19+]** If $G$ is an infinite, vertex-transitive, locally finite graph, then

$$\text{for all } \mu > 0, \quad \mathbb{P}\left[\text{oil-water model fixates}\right] = 1.$$

**Idea behind this fact**: Number of oil-water pairs at any fixed vertex is a *supermartingale*, thus it is reasonable that this number decreases in time and tends to zero, implying fixation. (Proof of this is far from obvious!)

# Facts and definitions

$B_L$:=ball of radius $L$ (large) centered at the origin.

- There is a 0-1 law for fixation;
- Abelian property is crucial.

We need to define

- Stopped Green's function $G_{B_L}(x, y) = \mathbb{E}[\#\text{visits to } y \text{ by SRW started at } x \text{ before exiting } B_L]$;
- HOLE:=vertex with no unpaired particles;
- Stabilization of $B_L$:=run process inside $B_L$, killing particles exiting it, until there are no firings anymore;
- GHOSTS: auxiliary particles (see below).

# Ghosts

Ghosts are auxiliary particles that do not interfere with oils nor waters.
A ghost is created at $x \in B_L$ whenever all the next requirements are satisfied

1. $x$ is a HOLE;
2. there is a firing next to $x$;
3. the water particle moves to $x$ and the oil particle does not.

Once created, ghosts perform simple random walk until they exit $B_L$, without seeing oils or waters.

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**Why ghosts?** Because #(ghosts+pairs visiting a fixed vertex) is a martingale! (Easier to control than supermartingale)

# Ghosts

**Crucial Lemma:** *Assume activity*. Then $\forall \varepsilon > 0$, $\forall M > 0$ there is $D = D(\varepsilon, M) > 0$ s.t.

$$\mathbb{P}\,[\#\text{ ghosts created at } x \text{ during stabilization of } B_L > M] > 1 - \varepsilon,$$

uniformly over all orderings of firings, and $\forall x \in B_L$ s.t.

$$d(x, B_L^c) > D.$$

(In words, if the system is active, it's very likely that *a lot* of ghosts will be created at $x$, for all $x$ far enough from the boundary of $B_L$.)

# Sketchy idea of the proof of the Theorem

By contradiction, assume activity.[1]

$$\begin{aligned}
&\mathbb{E}[\#\text{firings at } o \text{ in stabilizing } B_L] \\
&\quad = \mathbb{E}[\#(\text{pairs+ghosts}) \text{ visiting } o \text{ in stabil. of } B_L] \\
&\qquad - \mathbb{E}[\#\text{ghosts visiting } o \text{ in stabil. of } B_L] \\
&\quad \text{“} \le \text{''} \mu \sum_{y \in B_L} G_{B_L}(y,o) - \sum_{y \in B_L} \mathbb{E}[\#\text{ghosts created at } y] G_{B_L}(y,o) \\
&\quad \text{“} \overset{\text{crucial lemma}}{\le} \text{''} \mu \sum_{y \in B_L} G_{B_L}(y,o) - \sum_{y \in B_L} 2\mu G_{B_L}(y,o) < 0.
\end{aligned}$$

Since $(\#\text{firings at } o \text{ in stabilizing } B_L) \ge 0$, this is a contradiction.

[1] Inequalities “ $\le$ ” are roughly correct, modulo technical details.

Thank you for your attention!