Consensus and disagreement in opinion dynamics

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Goal: Understanding opinion dynamics, namely the long-time behaviour of certain interacting particle systems, where individuals/agents have opinions in some metric space, and tend to realign with each other.

Outline

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1. The Deffuant model
2. The compass model
3. Results on the compass model with $\theta = 1$ on $\mathbb{Z}$
4. Ingredients of the proof

Consider a connected and locally finite graph $G = (V, E)$: the vertices are interpreted as *individuals* or *agents* and two individuals can interact if they are linked by an edge. Individuals hold *opinions* in $[0, 1]$. We define a Markov process η_t with values in $[0, 1]^V$ where $\eta_t(v)_{v \in V}$ will denote the configuration of opinions at time t. Fix a

- *threshold parameter* $\theta \in [0, 1]$ and a
- *step parameter* $\mu \in (0, \frac{1}{2}]$.

All edges have exponential clocks. When the clock of $\langle u, v \rangle$ rings at time t and the current opinions are $\eta_{t-}(u) = a$ and $\eta_{t-}(v) = b$ there are two possibilities:

- If $|\eta_{t-}(u) - \eta_{t-}(v)| \leq \theta$, both agents will change their opinion by a step μ, i.e. $\eta_t(u) = a + \mu(b - a)$ and $\eta_t(v) = b + \mu(a - b)$.
- If $|\eta_{t-}(u) - \eta_{t-}(v)| > \theta$, then nothing happens (the agents do not trust each other since their opinions are too far away!).

There are different scenarios, depending on θ, μ and the initial configuration.

Example

If V is finite, and $\theta = 1$, the opinions will stabilize, i.e. $\eta_t(v) \to c$ where $c = \frac{1}{|V|} \sum_{v \in V} \eta_0(v)$.

(To see this, note that the arithmetic mean of the opinions does not change with the dynamics!)

Definition

We distinguish the following three asymptotic regimes:

(i) *No consensus*
There exist $\varepsilon > 0$ and two neighbors $\langle u, v \rangle$, s.t. for all $t_0 \geq 0$ there exists $t > t_0$ with

$$d\big(\eta_t(u), \eta_t(v)\big) \geq \varepsilon. \tag{1}$$

(ii) *Weak consensus*
Every pair of neighbors $\langle u, v \rangle$ will finally concur, i.e. for all $e = \langle u, v \rangle \in E$

$$d\big(\eta_t(u), \eta_t(v)\big) \to 0, \text{ as } t \to \infty. \tag{2}$$

(iii) *Strong consensus*
The value at every vertex converges to a common (possibly random) limit L, i.e. for all $v \in V$

$$d\big(\eta_t(v), L\big) \to 0, \text{ as } t \to \infty. \tag{3}$$

Definition

(continuation)
In cases (ii) and (iii), we speak of almost sure consensus / consensus in mean / consensus in probability whenever the convergence in (2) *and* (3) *is almost surely / in* $\mathcal{L}^1$ */ in probability.*

It is easy to show that on *finite* graphs, weak consensus implies strong consensus, hence the two notions are equivalent.

Theorem

Nicolas Lanchier 2011, Olle Häggström 2011
Consider the Deffuant model on $\mathbb{Z}$ with $\{\eta_0(v), v \in V\}$ iid with law $U[0,1]$. Fix $\mu \in (0, \frac{1}{2}]$. Then there are two regimes:

- *If $\theta < \frac{1}{2}$, then there is almost sure consensus and $\eta_t(v) \to \frac{1}{2}$ for $t \to \infty$.*
- *If $\theta > \frac{1}{2}$, there is no consensus.*

Later these results were extended beyond the uniform distribution on $[0,1]$ for the initial opinions, first to general univariate distributions by Olle Häggström and Timo Hirscher, then to vector-valued and measure-valued opinions by Timo Hirscher.

Conjecture

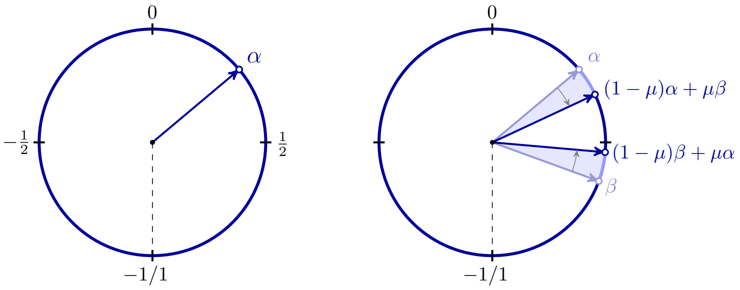
The theorem still holds true for the Deffuant model on $\mathbb{Z}^d$ for $d \geq 2$.

Olle Häggström and Timo Hirscher showed that in the Deffuant model on $\mathbb{Z}^d$ with $\theta = 1$, there is almost sure weak consensus. For $d \geq 2$, it is conjectured (but not proved!) that almost sure strong consensus holds.

Question

Can there be cases where almost sure weak consensus occurs, but no almost sure strong consensus?

We take an opinion space without "middle opinion", namely the unit circle S^1. The dynamics is defined in analogy to the Deffuant model.



We will parametrize $\mathcal{S} = S^1$ via the quotient space $\mathbb{R}/2\mathbb{Z}$, i.e.

$$\mathcal{S} = \{[x];\ -1 < x \leq 1\}, \quad \text{where } [x] = \{y \in \mathbb{R};\ \tfrac{y-x}{2} \in \mathbb{Z}\},$$

and define on it the canonical metric
$d([x],[y]) = \min\{|a-b|;\ a \in [x],\ b \in [y]\}$.

Indeed the compass model behaves quite differently than the Deffuant model with $\theta = 1$. We prove the following:

Theorem

For the compass model with $\theta = 1$ on $\mathbb{Z}$ with iid uniform initial distribution, there is weak consensus in mean, but no strong consensus in probability.

For $s \in \mathcal{S}$, denote by $\bar{s}$ the configuration which assigns the value s to all vertices, and let $\delta_{\bar{s}}$ denote the Dirac measure which assigns mass 1 to $\bar{s}$ and 0 to all other configurations.

Theorem

The set $\mathcal{I}$ of invariant measures for the compass model with $\theta = 1$ on $\mathbb{Z}$ is given by the convex hull of the set

$$\left\{\delta_{\bar{s}};\ s \in \mathcal{S}\right\}.$$

Remark

It is known that for the XY-model on $\mathbb{Z}$, there is a unique stationary distribution. See the beautiful book "Statistical mechanics on lattice systems" by Sacha Friedli and Yvan Velenik. This is in sharp contrast to our results.

No strong consensus:

Proposition

For the uniform compass model on $\mathbb{Z}$, there is no almost sure strong consensus.

This result readily follows from the symmetries of the model. Assume that there exists a $(-1,1]$-valued random variable L for which

$$d\big(\eta_t(v), L\big) \to 0, \text{ as } t \to \infty.$$

Then

$$B = \Big\{ \lim_{t\to\infty} d\big(\eta_t(v), L\big) = 0, \text{ for all } v \in \mathbb{Z} \Big\}$$

is an almost sure event and either $B \cap \{L \in (-1,0]\}$ or $B \cap \{L \in (0,1]\}$ has probability at least $\frac{1}{2}$. As we have complete rotational symmetry in $\mathcal{S}$, we can in fact conclude that these probabilities coincide, i.e. $P(B \cap \{L \in (-1,0]\}) = P(B \cap \{L \in (0,1]\}) = \frac{1}{2}$. Finally, the event $B \cap \{L \in [0,1)\}$ is invariant with respect to shifts on $\mathbb{Z}$, thus forced to either have probability 0 or 1, due to ergodicity of the model with respect to spatial shifts. Hence this leads to a contradiction.

Remark

The last argument goes through for the case $\theta < 1$.

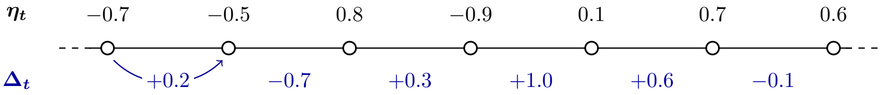
Weak consensus:

Proposition

The compass model on $\mathbb{Z}$ with uniform initial opinions exhibits weak consensus in mean.

The main ingredient to prove Proposition 4.2 is the following. Given a configuration of opinions $\eta_t = (\eta_t(v))_{v \in V} \in (-1,1]^V$, define the corresponding *configuration of edge differences* $\Delta_t = \big(\Delta_t(e)\big)_{e \in E}$ in the following way: Assign to each edge $e = \langle u, v \rangle$ the unique value $\Delta_t(e) \in (-1,1]$, such that

$$\eta_t(u) + \Delta_t(e) = \eta_t(v) \pmod{\mathcal{S}}.$$



Lemma

The function $t \mapsto \mathbb{E}\big|\Delta_t(e)\big|,\ t \in [0,\infty)$ is non-increasing.

Unfortunately our proof of the lemma relies on $d = 1$.

Thanks for your attention!