Renewal Approximation in the Online Increasing Subsequence Problem

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Ulam's problem

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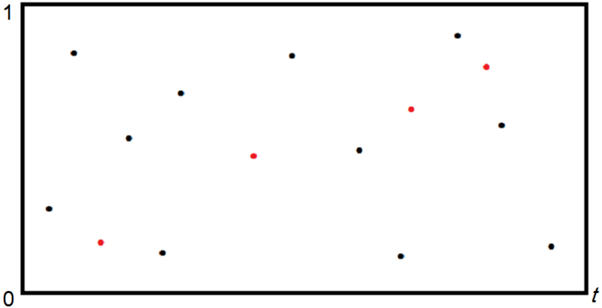
Ulam 1961: *What is the expected length of the longest increasing subsequence of a random permutation of t integers?*

Instead of permutation, one can consider the setting of i.i.d. random marks $X_1, \ldots, X_t$ sampled from a continuous distribution (let it be uniform-$[0, 1]$)

Hammersley 1972: Suppose the marks arrive by a Poisson process on $[0, t]$, so the sample size is random with Poisson(t)-distribution. A increasing subsequence $(x_1, s_2), \ldots, (x_k, s_k)$ of marks/arrival times is a chain in two dimensions: $x_1 < \cdots < x_k, \;\; s_1 < \cdots < s_k$



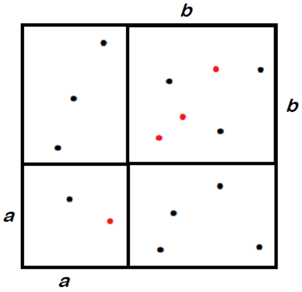
Hammersley: since only the area of rectangle matters, the maximum length $M(t)$ satisfies

$$M((a+b)^2) \geq M(a^2) + M(b^2),$$

hence by subadditivity

$$M(t) \sim c\sqrt{t}, \; t \to \infty$$

(in probability and in the mean).



Logan and Shepp 1977, Vershik and Kerov 1977:

$$\mathbb{E}M(t) \sim 2\sqrt{t}$$

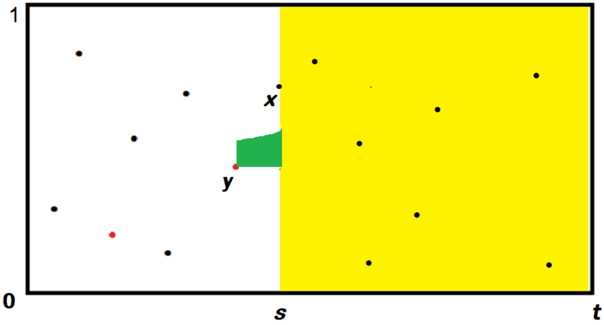
Baik, Deift and Johansson 1999:

$$\frac{M(t) - 2\sqrt{t}}{t^{1/6}} \xrightarrow{d} \text{Tracy–Widom distribution.}$$

D. Romik 2014: The Surprising Mathematics of Longest Increasing Subsequences.

The online selection problem

Samuels and Steele 1981: The marks are revealed to the observer one-by-one as they arrive. Each time a mark is observed, it can be selected or rejected, with decision becoming immediately final.
What is the maximum expected length, $v(t)$, of increasing subsequence which can be selected by a nonanticipating online strategy?



Subadditivity yields

$$v(t) \sim c\sqrt{t}$$

but gives no clue about the constant (of course, $c \leq 2$).

A selection strategy can be identified with a sequence of stopping times embedded in the Poisson process, such that the corresponding marks increase. For instance, the greedy strategy, selecting every consecutive *record* (i.e. a mark bigger than all seen so far), yields a sequence of expected length

$$\int_0^t \frac{1 - e^{-s}}{s} ds \sim \log t, \quad t \to \infty.$$

This is too far from optimality!

The principal asymptotics

Samuels and Steele 1981:

$$v(t) \sim \sqrt{2t}, \quad t \to \infty,$$

achieved by the strategy with *constant* acceptance window

$$0 < x - y < \sqrt{\frac{2}{t}},$$

where $(x, s) \in [0, 1] \times [0, t]$ is the current arrival, and y the last mark selected before time s.
Comparing with the *offline* asympotics $2\sqrt{t}$ in Ulam's problem, the factor $\sqrt{2}$ quantifies the advantage of a prophet over nonclairvoyant decision maker selecting in the real time.

The optimality equation

The maximal expected length satisfies the dynamic programming equation

$$v'(t) = \int_0^1 (v(t(1-x) + 1 - v(t))_+ \, dx, \qquad v(0) = 0.$$

Under the optimal strategy (x, s) is accepted iff

$$0 < \frac{x-y}{1-x} < \varphi^*((t-s)(1-x))$$

where y is the last selection and $\varphi^*(t)$ is the solution to

$$v(t(1-x)) + 1 - v(t) = 0$$

(for $t > v^{\leftarrow}(1) = 1.345\ldots$).
A strategy of this kind with some control function φ defining a variable acceptance window will be called *self-similar*.

The tightest known bounds

$$\sqrt{2t} - \log(1+\sqrt{2t}) + c_0 < v(t) < \sqrt{2t}$$

The upper bound: Baryshnikov and G 2000 by comparing with the bin-packing problem

$$\text{the expected number of choices} \to \max$$

subject to the ((mean value!) constraint that

$$\text{the } \textit{expectation of the sum} \text{ of selected marks} \leq 1.$$

In this problem the strategy choosing every $x < \sqrt{2/t}$ is exactly optimal.

The lower bound: Bruss and Delbaen 2001, using concavity of $v(t)$ and the optimality equation.

The asymptotic expansion

Let $L_\varphi(t)$ be the length of selected subsequence under the strategy with control function φ, in particular $v(t) = \mathbb{E} L_{\varphi^*}(t)$.

Theorem. *The expected length under the optimal strategy is*

$$v(t) \sim \sqrt{2t} - \frac{1}{12}\log t + c^* + \frac{\sqrt{2}}{144\sqrt{t}} + O(t^{-1})$$

and the variance is

$$\mathrm{Var}(L_{\varphi^*})(t) = \frac{\sqrt{2t}}{3} + \frac{1}{72}\log t + c_1 + O(t^{-1/2}\log t).$$

The optimal strategy is self-similar with

$$\varphi^*(t) \sim \sqrt{\frac{2}{t}} - \frac{1}{3t} + O(t^{-3/2}).$$

Constants c^*, c_1 are unknown.

Theorem *For every self-similar selection strategy with*

$$\varphi(t) = \sqrt{\frac{2}{t}} + O(t^{-1})$$

the expected length of increasing subsequence is within $O(1)$ *from the optimum, and the CLT holds*

$$\sqrt{3}\,\frac{L_\varphi(t) - \sqrt{2t}}{(2t)^{1/4}} \stackrel{d}{\to} \mathcal{N}(0,1).$$

Bruss and Delbaen 2004, Arlotto et al 2015 proved the CLT for the optimal strategy using concavity of $v(t)$ and martingale methods. Our approach relies on a renewal approximation to the 'remaining area process'.

Linearisation

With $z = \sqrt{t}$ as the size parameter and a change of variables, the equation for expected length under self-similar strategy becomes

$$u'(z) = 4 \int_0^1 (u(z-y) + 1 - u(z))_+ (1 - y/z) dy.$$

This is a special case of the renewal-type equation

$$u'_{r,\theta}(z) = 4 \int_0^{\theta(z)} (u_{r,\theta}(z-y) + r(z) - u_{r,\theta}(z))(1 - y/z) dy$$

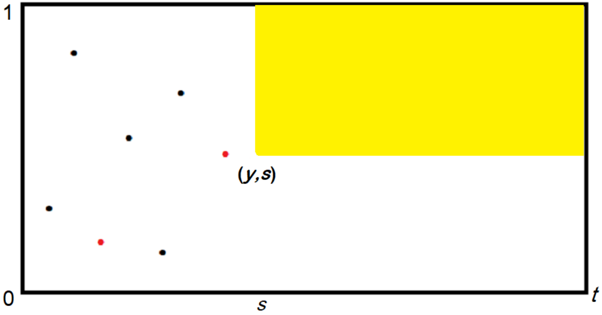
with given reward function $r(z)$ and control function $0 < \theta(z) \leq z$ related to a self-similar strategy via

$$\varphi(z^2) = 1 - \left(1 - \frac{\theta(z)}{z}\right)^2.$$

The admissible rectangle

Change of variables:

$$\text{the last so far selection } (y, s) \to z = \sqrt{(t-s)(1-y)}$$



Piecewise deterministic Markov process

For given control function $0 < \theta(z) \leq z$, a PDMP process Z on $[0, \infty)$ is defined by

(i) decreases with unit speed until absorption at 0,

(ii) jumps at probability rate $4\lambda(z)$, where

$$\lambda(z) := \theta(z) - \frac{\theta^2(z)}{2z},$$

(iii) if jumps, then from z to $z - y$, with y having density $(1 - y/z)/\lambda(z)$ for $y \in [0, \theta(z)]$.

The number of jumps $N_\theta(z)$ of Z starting from $z = \sqrt{t}$ is equal to $L_\varphi(t)$, the length of increasing subsequence under a self-similar strategy.

Asymptotic version of de Bruijn's method for DE's

The operator

$$\mathcal{I}_{\theta,r}g(z) := 4\int_0^{\theta(z)} (g(z-y)+r(z)-g(z))_+ (1-y/z)dy$$

has shift and monotonicity properties that imply

Lemma *If for large enough z,*

(a) $g'(z) > \mathcal{I}_{\theta,r}g(z)$ *then* $\limsup_{z\to\infty}(u_{\theta,r}(z)-g(z)) < \infty$,

(b) $g'(z) < \mathcal{I}_{\theta,r}g(z)$ *then* $\liminf_{z\to\infty}(u_{\theta,r}(z)-g(z)) > -\infty$.

Example For $g(z)=\alpha z$, in the optimality equation, (a) holds for $\alpha > \sqrt{2}$, and (b) holds for $\alpha < \sqrt{2}$, whence $u(z) \sim \sqrt{2}\,z$. Iterating twice ,

$$u(z) \sim \sqrt{2}\,z - \frac{1}{6}\log z + O(1), \quad z\to\infty.$$

But the method does not capture the $O(1)$-remainder.

Let $U(z_0, dz)$, be the occupation measure on $[0, z_0]$, for the sequence of jump points of Z starting from z_0, and controlled by the optimal $\theta^*(z)$. The density is

$$U(z_0, dz) = 4\lambda(z)p(z_0, z)dz,$$

where $p(z_0, z)$ is the probability that z is a drift point.
Lemma *There exists a pointwise limit* $p(z) := \lim_{z_0 \to \infty} p(z_0, z)$, *such that* $\lim_{z \to \infty} p(z) = 1/2$ *and for some* $a, b > 0$

$$|p(z_0, z) - p(z)| < ae^{-b(z_0 - z)}, \quad 0 < z < z_0.$$

The proof is by coupling: two independent Z-processes starting with z_1 and z_2 (where $z_1 < z_2$) with high probability visit the same drift point close to z_1.

The 'mean reward' for Z starting with $z > 0$ has representation

$$u_{\theta^*,r}(z) = \int_0^z r(y) U(z, dy).$$

Corollary *For integrable $r(z)$,*

$$u_{\theta^*,r}(z) \to \int_0^\infty r(y)\lambda(y)p(y)dy, \quad z \to \infty.$$

If $r(z) = O(z^{-\beta})$ with $\beta > 1$ then the convergence rate is $O(z^{-\beta+1})$.
This allows us to obtain the asymptotic expansions of the moments of $N_\theta(t)$ and of the length of selected sequence $L_\varphi(t)$ under self-similar strategies. In particular, $w(z) = (\mathbb{E}N_{\theta^*}(z))^2$ satisfies

$$w'(z) = 4\int_0^{\theta^*(z)} (w(z-y) - w(z) + (1 + 2u(z-y))(1 - y/z)dy,$$

$$w(0) = 0.$$

A renewal approximation to Z

The range of Z is an alternating sequence of drift intervals and gaps skipped by jumps. Let D_z be the size of generic drift interval and J_z that of jump. From

$$\theta^*(z) = \frac{1}{\sqrt{2}} + \frac{1}{12z} + O(z^{-2})$$

follows that for $z \to \infty$ that $4\lambda(z) \to 2\sqrt{2}$ and

$$D_z \xrightarrow{d} \frac{E}{2\sqrt{2}}, \quad J_z \xrightarrow{d} \frac{U}{\sqrt{2}},$$

where E and U are independent Exponential(1) and Uniform-$[0,1]$ random variables. At distance from 0, the generic jump of Z are approximable by decreasing renewal proces with cycle-size

$$D_z + J_{z-D_z} \xrightarrow{d} \frac{E}{2\sqrt{2}} + \frac{U}{\sqrt{2}} =: H$$

CLT by stochastic comparison

Cutsem and Ycart 1994, Haas and Miermont 2011, Alsmeyer and Marynych 2016: limit theorems for absorption times (or jump-counts) for decreasing Markov chains on $\mathbb{N}$.

Adapting the stochastic comparison method of Cutsem and Ycart, we squeeze

$$(1 + c/\underline{z})^{-1} H <_{\mathrm{st}} D_z + J_{z - D_z} <_{\mathrm{st}} (1 - c/\underline{z})^{-1} H$$

for $z > \underline{z}$, where $\underline{z} = \omega\sqrt{z}$ and ω large parameter. Accordingly, the number of jumps of Z within $[\underline{z}, z]$ is squeezed between two renewal processes which satisfy the CLT.

It is important that the cycle-size of Z is within $O(z^{-1})$ from the limit, by slower convergence rate $O(z^{-1/2+\epsilon})$ the normal approximation may fail.

Fluctuations of the shape of selected increasing sequence

$Y(s)$ the last mark selected by the optimal strategy by time $s \in [0, t]$.

Theorem For $t \to \infty$

$$(t^{1/4}(Y(\tau t) - \tau))_{\tau \in [0,1]} \Rightarrow \text{Brownian bridge}$$

in the Skorohod topology on $D[0, 1]$.

Longest chains in $d+1$ dimensions

$(x_1, s_1), \dots, (x_k, s_k) \in [0,1]^d \times [0,t]$ is a chain if the sequence increases in every component.

Ulam's problem: $M(t) \sim ct^{1/(d+1)}$, but the constant is unknown (estimates in Bollobas and Winkler 1988).
Online chains in d dimensions. Our methods extend to the online increasing subsequence problem with marks sampled from uniform distribution in $[0,1]^d$. The principal asymptotics is

$$v(t) \sim \frac{(d+1)}{\{(d+1)!\}^{1/(d+1)}} \; t^{1/(d+1)}$$

Bruss and Delbaen 2004: prove a more complex functional Gaussian limit for a random centering.