On semigroups associated with the Dunkl operators

Joint work with Jacek Dziubański

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- Generator: $\Delta = \sum_{j=1}^{N} \partial_j^2$
- Associated multiplier: $e^{-|\xi|^2}$

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Upper integral kernel estimate (t=1)

$$Ce^{-c|x-y|^{\frac{2\ell}{2\ell-1}}}$$

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Dunkl heat semigroup

- Generator: $\Delta = \sum_{j=1}^{N} T_j^2$
- Associated multiplier: $e^{-|\xi|^2}$

Upper heat kernel estimate ($t = 1$)

$$w(B(\mathbf{x},1))^{-1} e^{-cd(\mathbf{x},\mathbf{y})^2}$$

Semigroups associated with higher order Dunkl operators

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Theorem (Hörmander)

Let ψ be a smooth radial function such that $\operatorname{supp}\psi \subseteq \{\xi : \frac{1}{4} \leqslant \|\xi\| \leqslant 4\}$ and $\psi(\xi) \equiv 1$ for $\{\xi : \frac{1}{2} \leqslant \|\xi\| \leqslant 2\}$. If m satisfies

$$M = \sup_{t>0} \|\psi(\cdot)m(t\cdot)\|_{W_2^s} < \infty$$

for some $s > \mathbf{N}/2$, then

$$\widehat{\mathcal{T}_m f} = (m\hat{f}),$$

is

(A) of weak type $(1,1)$,

(B) of strong type (p,p) for $1 < p < \infty$,

(C) bounded on the Hardy space H^1_{atom}.

Theorem (J. Dziubański, A.H.)

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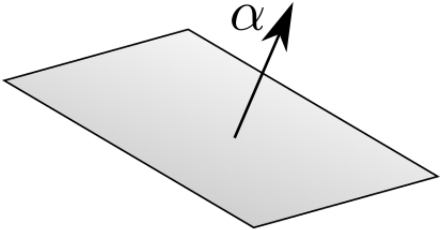
Reflections

We consider the Euclidean space $\mathbb{R}^N$ with the scalar product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^N x_j y_j$, $\mathbf{x} = (x_1, \dots, x_N)$, $\mathbf{y} = (y_1, \dots, y_N)$.

Reflection

For a nonzero vector $\alpha \in \mathbb{R}^N$ the reflection σ_α with respect to the orthogonal hyperplane $\alpha^\perp$ orthogonal to a nonzero vector α is given by

$$\sigma_\alpha \mathbf{x} = \mathbf{x} - 2\frac{\langle \mathbf{x}, \alpha \rangle}{\|\alpha\|^2}\alpha.$$



Root system and Weyl group

Root system

A finite set $R \subset \mathbb{R}^N \setminus \{0\}$ is called a *root system* if $\sigma_\alpha(R) = R$ for every $\alpha \in R$.

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Weyl group

The finite group G generated by the reflections σ_α is called the *Weyl group* (*reflection group*) of the root system.

Examples - product root systems

A_1

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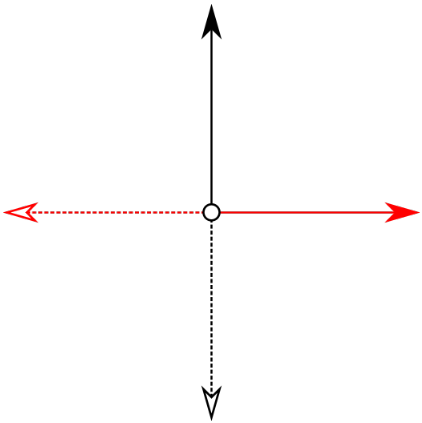
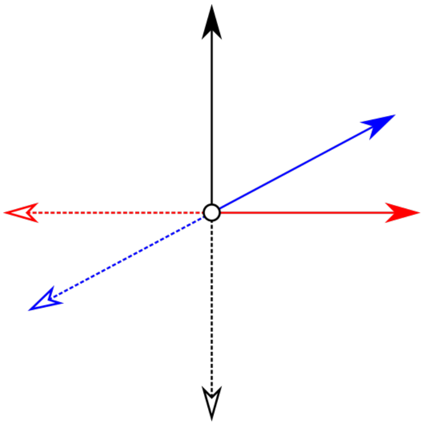


$A_1 \times A_1$



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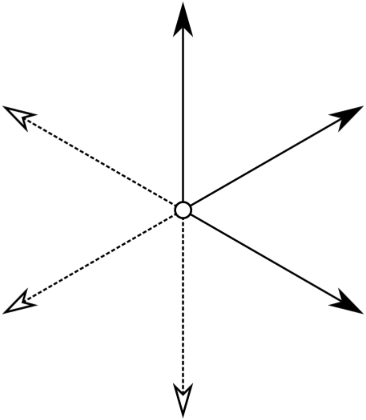
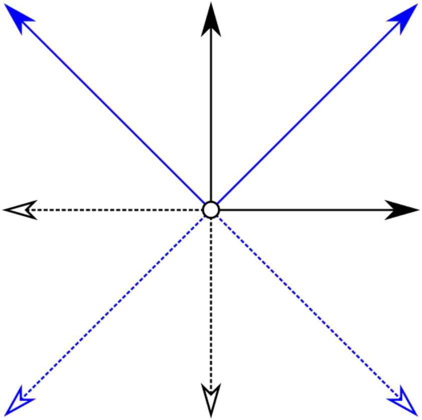
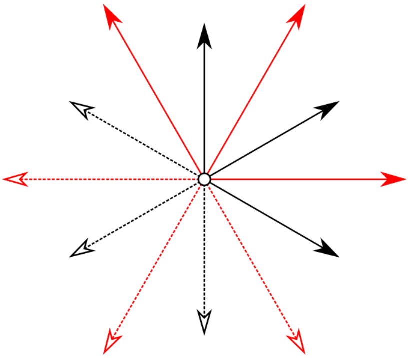
Examples of root systems



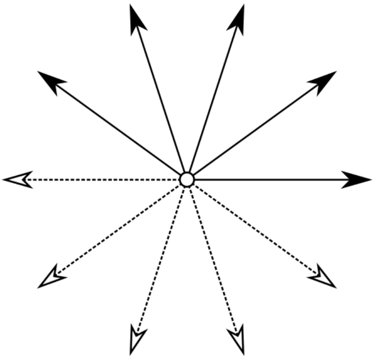
A_2



B_2



G_2



$I_2(5)$

Multiplicity function

A *multiplicity function* is a G-invariant function $k : R \to \mathbb{C}$ which will be fixed and $\geqslant 0$.

Let

$$\mathbf{N} = N + \sum_{\alpha \in R} k(\alpha)$$

($\mathbf{N}$ is the **homogeneous dimension**).

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(**N** is the **homogeneous dimension**). We define the measure

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We have

$$w(B(\mathbf{x}, r)) \sim r^{\mathbf{N}} \prod_{\alpha \in R} (|\langle \mathbf{x}, \alpha \rangle| + r)^{k(\alpha)},$$

so $dw(\mathbf{x})$ is doubling.

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Dunkl operators

Given a root system R and multiplicity function $k(\alpha)$ the *Dunkl operator* T_ξ is the following k-deformation of the directional derivative ∂_ξ by a difference operator:

$$T_\xi f(\mathbf{x}) = \partial_\xi f(\mathbf{x})$$

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Example for $N = 1$

$$Tf(x) = \partial f(x) + k(\alpha) \frac{f(x) - f(-x)}{x}.$$

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Difference

No Leibniz rule!

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Dunkl kernel

For fixed $\mathbf{y} \in \mathbb{R}^N$ the *Dunkl kernel* $E(\mathbf{x}, \mathbf{y})$ is the unique solution of the system

$$T_\xi f = \langle \xi, \mathbf{y} \rangle f, \quad f(0) = 1.$$

In particular,

$$T_{j,\mathbf{x}} E(\mathbf{x}, \mathbf{y}) = T_{e_j,\mathbf{x}} E(\mathbf{x}, \mathbf{y}) = y_j E(\mathbf{x}, \mathbf{y}).$$

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Dunkl transform =generalization of Fourier transform

The *Dunkl transform* is defined on $L^1(dw)$ by

$$\mathcal{F} f(\xi) = c_k^{-1} \int_{\mathbb{R}^N} f(\mathbf{x}) E(\mathbf{x}, -i\xi) \, dw(\mathbf{x}).$$

1. Introduction
   - Fourier analysis in the rational Dunkl setting

2. Dunkl translations

3. Semigroups of operators
   - Radial case - heat semigroup associated with the Dunkl Laplacian
   - Nonradial cases

4. Idea of the proofs
   - Convolution with radial function
   - Support of $\tau_{\mathbf{x}} f(-\cdot)$

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Dunkl translations =generalization of translations

The *Dunkl translation* $\tau_{\mathbf{x}} f$ of $f \in \mathcal{S}(\mathbb{R}^N)$ by $\mathbf{x} \in \mathbb{R}^N$ is defined by

$$\tau_{\mathbf{x}} f(\mathbf{y}) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, \mathbf{x})\, E(i\xi, \mathbf{y})\, \mathcal{F}f(\xi)\, dw(\xi).$$

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Dunkl convolution =generalization of convolution

The *Dunkl convolution* of two reasonable functions is defined by

$$(f * g)(\mathbf{x}) = c_k\, \mathcal{F}^{-1}[(\mathcal{F}f)(\mathcal{F}g)](\mathbf{x}) = \int_{\mathbb{R}^N} \tau_{\mathbf{x}}f(-\mathbf{y})g(\mathbf{y})\, dw(\mathbf{y}).$$

Essential difference

Dunkl translations don't form a group

$$\tau_{\mathbf{x}}\tau_{\mathbf{y}} \neq \tau_{\mathbf{x}+\mathbf{y}}$$

Positivity of the Dunkl translations

Positivity of the Dunkl translations (radial function)

Suppose that $f \in L^2(dw)$ is **radial** and $f \geqslant 0$ a.e. Then $\tau_{\mathbf{x}} f \geqslant 0$ a. e. for all $\mathbf{x} \in \mathbb{R}^N$.

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Nonpositivity of the Dunkl translations

There are: root system R, multiplicity function $k \geqslant 0$, $\mathbf{x} \in \mathbb{R}^N$, and $L^2(dw) \ni f \geqslant 0$ a.e. such that $\tau_{\mathbf{x}} f < 0$ on the set of positive Lebesgue measure.

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$L^2(dw)$-case

By the Plancherel's theorem for the Dunkl transform

$$\sup_{\mathbf{x}\in\mathbb{R}^N} \left\|\tau_{\mathbf{x}}\right\|_{L^2(dw)\mapsto L^2(dw)} = 1.$$

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By the Plancherel's theorem for the Dunkl transform

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Open problem

Let $1 < p < \infty$. Then

$$\sup_{\mathbf{x}\in\mathbb{R}^N} \|\tau_{\mathbf{x}}\|_{L^p(dw)\mapsto L^p(dw)} < \infty.$$

Radial case

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Exception

If $f \in L^p(dw)$ is **radial**, then for all $\mathbf{x} \in \mathbb{R}^N$ we have

$$\|\tau_{\mathbf{x}} f\|_{L^p(dw)} \leqslant \|f\|_{L^p(dw)}.$$

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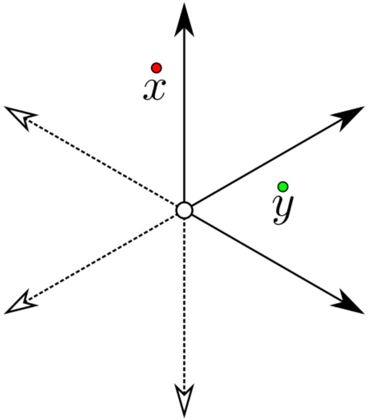
Conclusion: Dunkl translation of radial function is easier to treat.

Let us define the distance of the orbits $\mathcal{O}(\mathbf{x})$ and $\mathcal{O}(\mathbf{y})$ to be

$$d(\mathbf{x},\mathbf{y}) = \min_{\sigma \in G} \|\sigma(\mathbf{x}) - \mathbf{y}\|.$$



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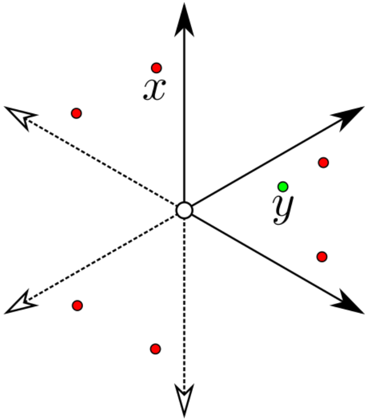

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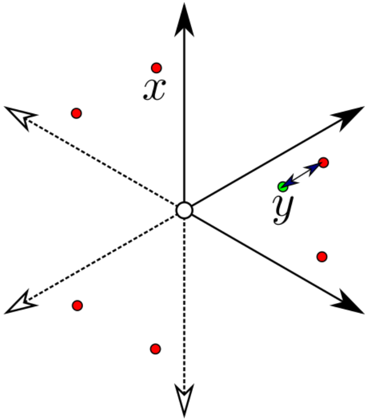

Dunkl Laplacian

The *Dunkl Laplacian* associated with G and k is the differential-difference operator

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Heat semigroup

The operator Δ generates the semigroup $H_t = e^{t\Delta}$ of linear self-adjoint contractions on $L^2(dw)$. The semigroup has the form

$$e^{t\Delta} f(\mathbf{x}) = \int_{\mathbb{R}^N} \tau_{\mathbf{x}} h_t(-\mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y}),$$

where h_t is classical heat kernel, which is **radial**.

Theorem (J.-P.. Anker, J. Dziubański, A.H.)

There are $C, c > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$ we have

$$h_t(\mathbf{x}, \mathbf{y}) \leqslant C\, w(B(\mathbf{x}, \sqrt{t}\,))^{-1}\, e^{-c\, d(\mathbf{x},\mathbf{y})^2/t},$$

$$|h_t(\mathbf{x}, \mathbf{y}) - h_t(\mathbf{x}, \mathbf{y}')| \leqslant C \left(\frac{\|\mathbf{y}-\mathbf{y}'\|}{\sqrt{t}}\right) w(B(\mathbf{x}, \sqrt{t}\,))^{-1}\, e^{-c\, d(\mathbf{x},\mathbf{y})^2/t}.$$

Derivatives of higher order

Semigroup operator associated with Dunkl derivatives of higher order

We define

$$L = (-1)^{\ell+1} \sum_{j=1}^{N} T_j^{2\ell}.$$

It is generator of semigroup of operators on $L^2(dw)$ with kernels of the form $q_t(\mathbf{x}, \mathbf{y}) = \tau_{\mathbf{x}}(\mathcal{F}q)_t(-\mathbf{y})$, where q is the associated **nonradial** multiplier.

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Theorem (J. Dziubański, A.H.)

There are constants $C, c > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ we have

$$|q_1(\mathbf{x}, \mathbf{y})| \leqslant C(w(B(\mathbf{x}, 1)))^{-1} \exp(-cd(\mathbf{x}, \mathbf{y})^{2\ell/(2\ell-1)}).$$

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Hörmander's multiplier theorem

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Hörmander's multiplier theorem

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so

$$\tau_{\mathbf{x}}(\mathcal{F}q)(-\mathbf{y}) = \tau_{\mathbf{x}}(\mathcal{F}q^{(\varepsilon)} * h_\varepsilon)(-\mathbf{y}) = (\mathcal{F}q^{(\varepsilon)}) * \tau_{\mathbf{x}}(h_\varepsilon)(-\mathbf{y}).$$

Translation is on radial function!

g-radial, f - not necessary radial

Further problem

g-radial, f - not necessary radial

Classical case (trivial)

$$\|\tau_{\mathbf{x}}(f * g)\|_{L^1(dw)} \leqslant \|f\|_{L^1(dw)} \|g\|_{L^\infty}$$

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g-radial, f - not necessary radial

Classical case (trivial)

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Dunkl case (no L^1-boundedness!)

$$\|\tau_{\mathbf{x}}(f * g)\|_{L^1(dw)} \leqslant \|f(\cdot)(1+|\cdot|)^{\mathbf{N}/2}\|_{L^1(dw)}\|g(\cdot)(1+|\cdot|)^{\mathbf{N}}\|_{L^\infty}$$

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g-radial, f - not necessary radial

Classical case (trivial)

$$\|\tau_{\mathbf{x}}(f*g)\|_{L^1(dw)} \leqslant \|f\|_{L^1(dw)}\|g\|_{L^\infty}$$

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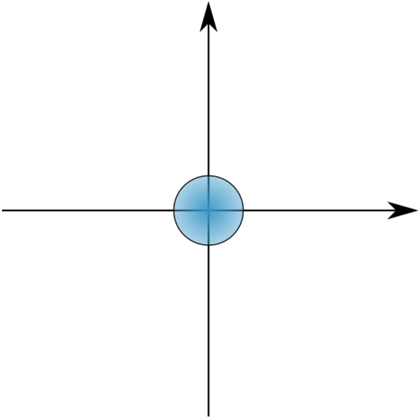
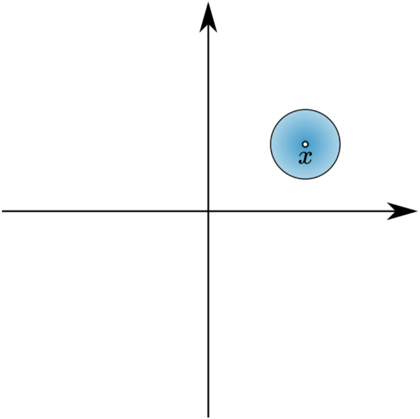
Idea: Pass to L^2 by Cauchy–Schwarz.

$$\|\tau_{\mathbf{x}}(f*g)\|_{L^1(dw)} \leqslant w(\operatorname{supp}\tau_{\mathbf{x}}(f*g))^{1/2}\|\tau_{\mathbf{x}}(f*g)\|_{L^2(dw)}$$

Suppose that $f \in L^2(dw)$ is such that $\operatorname{supp} f \subseteq B(0,1)$.



Suppose that $f \in L^2(dw)$ is such that $\operatorname{supp} f \subseteq B(0,1)$.
If we consider $f_{\mathbf{x}} = f(\mathbf{x} - \cdot)$, then $\operatorname{supp} f_{\mathbf{x}} \subseteq B(\mathbf{x},1)$.



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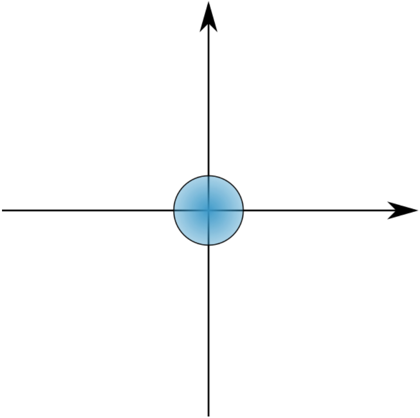
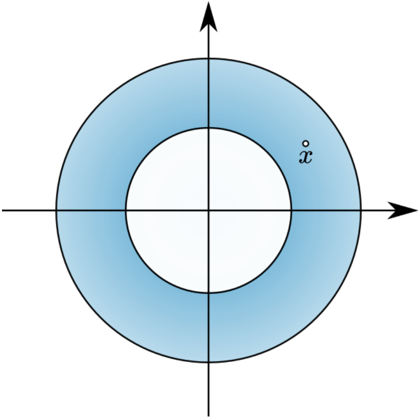
If we consider $f_{\mathbf{x}} = f(\mathbf{x} - \cdot)$, then $\operatorname{supp} f_{\mathbf{x}} \subseteq B(\mathbf{x}, 1)$.
Question: What about $\operatorname{supp} \tau_{\mathbf{x}} f(-\cdot)$?



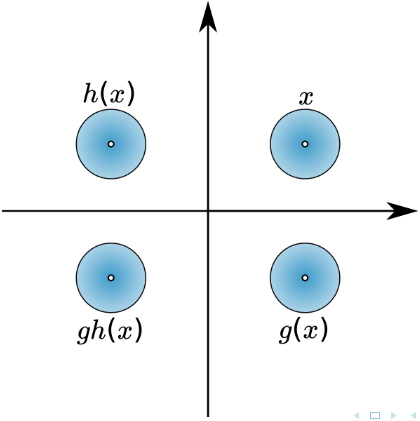
Results of Amri, Anker and Sifi (Paley-Wiener approach) imply

$$\operatorname{supp} \tau_{\mathbf{x}} f(-\cdot) \subseteq \{\mathbf{y} : \|\mathbf{x}\| - 1 \leqslant \|\mathbf{y}\| \leqslant \|\mathbf{x}\| + 1\}.$$



Results of Rösler imply that if f is **radial**, then

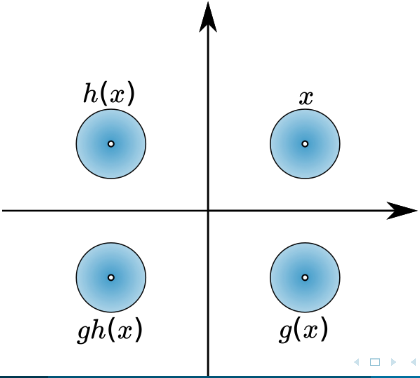
$$\operatorname{supp} \tau_{\mathbf{x}} f(- \cdot) \subseteq \mathcal{O}(B(\mathbf{x},1)) = \bigcup_{g \in G} B(g(\mathbf{x}),1).$$



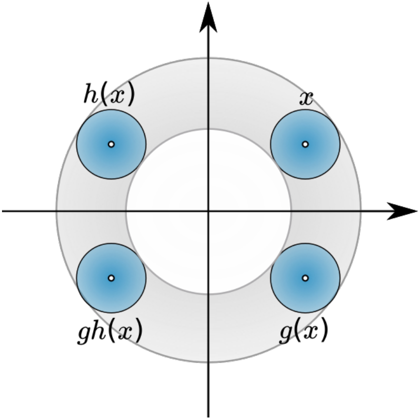
Theorem (J. Dziubański, A.H.)

Let $f \in L^2(dw)$, $\operatorname{supp} f \subseteq B(0,1)$, and $\mathbf{x} \in \mathbb{R}^N$. Then

$$\operatorname{supp} \tau_{\mathbf{x}} f(-\,\cdot) \subseteq \mathcal{O}(B(\mathbf{x},1)).$$



The measure of $\mathcal{O}(B(\mathbf{x},1))$ is **much smaller** than the measure of $\{\mathbf{y} : \|\mathbf{x}\| - 1 \leqslant \|\mathbf{y}\| \leqslant \|\mathbf{x}\| + 1\}$.



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$$g_L(\mathbf{x}) = \max\{0, (1 - \|\mathbf{x}\|^2)\}^L.$$

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For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{N}_0^N = (\mathbb{N} \cup \{0\})^N$ we define

$$T_j^0 = I, \quad T^\alpha := T_1^{\alpha_1} \circ T_2^{\alpha_2} \circ \ldots \circ T_N^{\alpha_N}.$$

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1. By induction we show that if p is a polynomial of degree d, then $p(\mathbf{x})g_L(\mathbf{x})$ can be written as

$$p(\mathbf{x})g_L(\mathbf{x}) = \sum_{\ell=0}^{d} \sum_{\|\alpha\| \leqslant \ell} c_{\ell,\alpha} T^\alpha(g_{L+\ell})(\mathbf{x}) \text{ for some } c_{\ell,\alpha} \in \mathbb{C}.$$

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The key point is the fact that the **Leibniz rule** can be applied

$$T_j(pg_L) = g_L(T_j p) + p(T_j g_L).$$

2. The set $\{p(\cdot)g_1(\cdot) : p \text{ is a polynomial}\}$ is dense in $L^2(B(0,1), dw)$.

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3. $\tau_{\mathbf{x}}$ is a contraction on $L^2(dw)$, so for any $\varepsilon > 0$ there is a polynomial p such that

$$\|\tau_{\mathbf{x}} f - \tau_{\mathbf{x}}(pg_1)\|_{L^2(dw)} \leqslant \|f - pg_1\|_{L^2(dw)} < \varepsilon. \qquad (\star)$$

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4. The Dunkl translations commute with the Dunkl operators, so

$$\tau_{\mathbf{x}}(pg_1)(-\mathbf{y}) = \sum_{\ell=0}^{d} \sum_{\|\alpha\| \leqslant \ell} c_{\ell,\alpha} T^{\alpha} \tau_{\mathbf{x}}(g_{1+\ell})(-\mathbf{y}).$$

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5. By the results of Rösler $\operatorname{supp}\ T^{\alpha}\tau_{\mathbf{x}}(g_{1+\ell}) \subseteq \mathcal{O}(B(\mathbf{x},1))$, so $(\star)$ implies the claim.

Thank you for your attention.

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