On semigroups associated with the Dunkl operators
Joint work with Jacek Dziubański

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1. Introduction
   - Fourier analysis in the rational Dunkl setting

2. Dunkl translations

3. Semigroups of operators
   - Radial case - heat semigroup associated with the Dunkl Laplacian
   - Nonradial cases

4. Idea of the proofs
   - Convolution with radial function
   - Support of $\tau_x f(\cdot)$
Some classical semigroups

Classical heat semigroup

- Generator: $\Delta = \sum_{j=1}^{N} \partial_j^2$
- Associated multiplier: $e^{-|\xi|^2}$
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**Upper heat kernel estimate ($t = 1$)**

$$\frac{1}{(4\pi)^{N/2}} e^{-\frac{1}{4}|x-y|^2}$$
Some classical semigroups

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- Associated multiplier: \( e^{-|\xi|^2} \)

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\frac{1}{(4\pi)^{N/2}} e^{-\frac{1}{4}|x-y|^2}
\]

Semigroups associated with higher order derivatives

- Generator:
  \[
  L = (-1)^{\ell+1} \sum_{j=1}^{N} \partial_j^{2\ell}
  \]
- Associated multiplier:
  \[
  e^{-\sum_{j=1}^{N} |\xi_j|^{2\ell}}
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### Classical heat semigroup

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### Semigroups associated with higher order derivatives

- **Generator:** 
  \( L = (-1)^{\ell + 1} \sum_{j=1}^{N} \partial_{j}^{2\ell} \)
- **Associated multiplier:** 
  \[ e^{-\sum_{j=1}^{N} |\xi_{j}|^{2\ell}} \]

### Upper integral kernel estimate \( (t = 1) \)

\[
Ce^{-c|x-y|^{\frac{2\ell}{2\ell-1}}}
\]
Some Dunkl semigroups

**Dunkl heat semigroup**
- Generator: $\Delta = \sum_{j=1}^{N} T_j^2$
- Associated multiplier: $e^{-|\xi|^2}$

Upper heat kernel estimate ($t = 1$)

$$w(B(x, 1))^{-1} e^{-cd(x,y)^2}$$

**Semigroups associated with higher order Dunkl operators**
- Generator: $L = (-1)^{\ell + 1} \sum_{j=1}^{N} T_j^{2\ell}$
- Associated multiplier: $e^{-\sum_{j=1}^{N} |\xi_j|^{2\ell}}$

Upper integral kernel estimate ($t=1$)

$$w(B(x, 1))^{-1} e^{-cd(x,y)^{\frac{2\ell}{2\ell-1}}}$$
Hörmander’s multiplier theorem

Theorem (Hörmander)

Let \( \psi \) be a smooth radial function such that \( \text{supp} \, \psi \subseteq \{ \xi : \frac{1}{4} \leq \| \xi \| \leq 4 \} \) and \( \psi(\xi) \equiv 1 \) for \( \{ \xi : \frac{1}{2} \leq \| \xi \| \leq 2 \} \). If \( m \) satisfies

\[
M = \sup_{t > 0} \| \psi(\cdot)m(t\cdot) \|_{W^s_2} < \infty
\]

for some \( s > \frac{N}{2} \), then

\[
\hat{T}_m f = (m\hat{f}),
\]

is

(A) of weak type \((1,1)\),

(B) of strong type \((p,p)\) for \(1 < p < \infty\),

(C) bounded on the Hardy space \( H^1_{\text{atom}} \).
Hörmander’s multiplier theorem

Theorem (J. Dziubański, A.H.)

Let \( \psi \) be a smooth radial function such that \( \text{supp} \psi \subseteq \{ \xi : \frac{1}{4} \leq \| \xi \| \leq 4 \} \) and \( \psi(\xi) \equiv 1 \) for \( \{ \xi : \frac{1}{2} \leq \| \xi \| \leq 2 \} \). If \( m \) satisfies

\[
M = \sup_{t > 0} \| \psi(\cdot)m(t\cdot) \|_{W_2^s} < \infty
\]

for some \( s > N \), then

\[
\mathcal{I}_m f = \mathcal{F}^{-1}(m\mathcal{F}f),
\]

is

(A) of weak type \((1, 1)\),

(B) of strong type \((p, p)\) for \( 1 < p < \infty \),

(C) bounded on the Hardy space \( H^1_{\text{atom}} \).
We consider the Euclidean space $\mathbb{R}^N$ with the scalar product $\langle x, y \rangle = \sum_{j=1}^{N} x_j y_j$, $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$.

**Reflection**

For a nonzero vector $\alpha \in \mathbb{R}^N$ the reflection $\sigma_\alpha$ with respect to the orthogonal hyperplane $\alpha^\perp$ orthogonal to a nonzero vector $\alpha$ is given by

$$\sigma_\alpha x = x - 2\frac{\langle x, \alpha \rangle}{\|\alpha\|^2} \alpha.$$
Root system

A finite set \( R \subset \mathbb{R}^N \setminus \{0\} \) is called a root system if \( \sigma_\alpha(R) = R \) for every \( \alpha \in R \).
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Weyl group

The finite group $G$ generated by the reflections $\sigma_\alpha$ is called the Weyl group (reflection group) of the root system.
Examples - product root systems

$A_1$
Examples - product root systems

$A_1 \times A_1$

$A_1 \times A_1 \times A_1$
Examples of root systems

\[ A_2 \quad B_2 \]
Examples of root systems

$G_2$

$I_2(5)$
A multiplicity function is a $G$-invariant function $k : R \to \mathbb{C}$ which will be fixed and $\geq 0$. 
Let
\[ N = N + \sum_{\alpha \in R} k(\alpha) \]

\((N \text{ is the } \text{homogeneous dimension}).\)
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(N is the **homogeneous dimension**). We define the measure

\[ w(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k(\alpha)}. \]
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(N is the **homogeneous dimension**). We define the measure

\[ w(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k(\alpha)}. \]

We have

\[ w(B(x, r)) \sim r^N \prod_{\alpha \in R} (|\langle x, \alpha \rangle| + r)^{k(\alpha)}, \]

so \( dw(x) \) is doubling.
Dunkl operators

Given a root system $R$ and multiplicity function $k(\alpha)$ the Dunkl operator $T_\xi$ is the following $k$-deformation of the directional derivative $\partial_\xi$ by a difference operator:

$$T_\xi f(x) = \partial_\xi f(x)$$
Dunkl operators

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$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$
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Example for $N = 1$

$$T f(x) = \partial f(x) + k(\alpha) \frac{f(x) - f(-x)}{x}.$$
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Example for $N = 1$

$$Tf(x) = \partial f(x) + k(\alpha) \frac{f(x) - f(-x)}{x}.$$ 

Difference

No Leibniz rule!
Dunkl kernel

For fixed \( y \in \mathbb{R}^N \) the \textit{Dunkl kernel} \( E(x, y) \) is the unique solution of the system

\[
T_\xi f = \langle \xi, y \rangle f, \quad f(0) = 1.
\]

In particular,

\[
T_{j, x} E(x, y) = T_{e_j, x} E(x, y) = y_j E(x, y).
\]
Dunkl kernel

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\( E(x, y) \) is a generalization of \( \exp(\langle x, y \rangle) \).
For fixed $y \in \mathbb{R}^N$ the Dunkl kernel $E(x, y)$ is the unique solution of the system

$$T_\xi f = \langle \xi, y \rangle f, \quad f(0) = 1.$$ 

In particular,

$$T_{j,x}E(x, y) = Te_{j,x}E(x, y) = y_j E(x, y).$$

$E(x, y)$ is a generalization of $\exp(\langle x, y \rangle)$.

The Dunkl transform is defined on $L^1(dw)$ by

$$\mathcal{F}f(\xi) = c_k^{-1} \int_{\mathbb{R}^N} f(x)E(x, -i\xi) \, dw(x).$$
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Dunkl translations \(= \text{generalization of translations}\)

The Dunkl translation \(\tau_x f\) of \(f \in S(\mathbb{R}^N)\) by \(x \in \mathbb{R}^N\) is defined by

\[
\tau_x f(y) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, x) E(i\xi, y) \mathcal{F}f(\xi) \, dw(\xi).
\]
Dunkl translations = generalization of translations

The *Dunkl translation* \( \tau_x f \) of \( f \in S(\mathbb{R}^N) \) by \( x \in \mathbb{R}^N \) is defined by

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\]

Dunkl convolution = generalization of convolution

The *Dunkl convolution* of two reasonable functions is defined by

\[
(f \ast g)(x) = c_k \mathcal{F}^{-1}[(\mathcal{F}f)(\mathcal{F}g)](x) = \int_{\mathbb{R}^N} \tau_x f(-y)g(y) \, dw(y).
\]
Dunkl translations don’t form a group

\[ \tau_x \tau_y \neq \tau_{x+y} \]
Positivity of the Dunkl translations (radial function)

Suppose that $f \in L^2(dw)$ is radial and $f \geq 0$ a.e. Then $\tau_x f \geq 0$ a.e. for all $x \in \mathbb{R}^N$. 

Nonpositivity of the Dunkl translations

There are: root system $R$, multiplicity function $k$, $x \in \mathbb{R}^N$, and $L^2(dw) \ni f < 0$ a.e. such that $\tau_x f < 0$ on the set of positive Lebesgue measure.
Positivity of the Dunkl translations (radial function)

Suppose that $f \in L^2(dw)$ is radial and $f \geq 0$ a.e. Then $\tau_x f \geq 0$ a.e. for all $x \in \mathbb{R}^N$.

Nonpositivity of the Dunkl translations

There are: root system $R$, multiplicity function $k \geq 0$, $x \in \mathbb{R}^N$, and $L^2(dw) \ni f \geq 0$ a.e. such that $\tau_x f < 0$ on the set of positive Lebesgue measure.
**$L^2(dw)$-case**

By the Plancherel’s theorem for the Dunkl transform

$$\sup_{x \in \mathbb{R}^N} \| \tau_x \|_{L^2(dw) \rightarrow L^2(dw)} = 1.$$
Boundedness of the Dunkl translations

$L^2(dw)$-case

By the Plancherel’s theorem for the Dunkl transform

$$\sup_{x \in \mathbb{R}^N} \| \tau_x \|_{L^2(dw) \rightarrow L^2(dw)} = 1.$$  

Open problem

Let $1 < p < \infty$. Then

$$\sup_{x \in \mathbb{R}^N} \| \tau_x \|_{L^p(dw) \rightarrow L^p(dw)} < \infty.$$
If \( f \in L^p(dw) \) is radial, then for all \( x \in \mathbb{R}^N \) we have

\[
\| \tau_x f \|_{L^p(dw)} \leq \| f \|_{L^p(dw)}.
\]

Conclusion: Dunkl translation of radial function is easier to treat.
Radial case

**Exception**

If \( f \in L^p(dw) \) is radial, then for all \( x \in \mathbb{R}^N \) we have

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**Conclusion:** Dunkl translation of radial function is easier to treat.
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Let us define the distance of the orbits $\mathcal{O}(x)$ and $\mathcal{O}(y)$ to be

$$d(x, y) = \min_{\sigma \in G} \|\sigma(x) - y\|.$$
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$$d(x, y) = \min_{\sigma \in G} \| \sigma(x) - y \| \leq \| x - y \|.$$
**Dunkl Laplacian**

The *Dunkl Laplacian* associated with $G$ and $k$ is the differential-difference operator

$$\Delta = \sum_{j=1}^{N} T_j^2.$$
Dunkl Laplacian

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Heat semigroup

The operator $\Delta$ generates the semigroup $H_t = e^{t\Delta}$ of linear self-adjoint contractions on $L^2(dw)$. The semigroup has the form

$$e^{t\Delta}f(x) = \int_{\mathbb{R}^N} \tau_x h_t(-y)f(y) \, dw(y),$$

where $h_t$ is classical heat kernel, which is radial.
Theorem (J.-P. Anker, J. Dziubański, A.H.)

There are $C, c > 0$ such that for all $x, y \in \mathbb{R}^N$ and $t > 0$ we have

$$h_t(x, y) \leq C w(B(x, \sqrt{t}))^{-1} e^{-c d(x,y)^2/t},$$

$$|h_t(x, y) - h_t(x, y')| \leq C \left( \frac{||y - y'||}{\sqrt{t}} \right) w(B(x, \sqrt{t}))^{-1} e^{-c d(x,y)^2/t}.$$
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Semigroup operator associated with Dunkl derivatives of higher order

We define

\[ L = (-1)^{\ell+1} \sum_{j=1}^{N} T_j^{2\ell}. \]

It is generator of semigroup of operators on \( L^2(dw) \) with kernels of the form \( q_t(x, y) = \tau_x(Fq)_t(-y) \), where \( q \) is the associated nonradial multiplier.
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Theorem (J. Dziubański, A.H.)

There are constants \( C, c > 0 \) such that for all \( x, y \in \mathbb{R}^N \) we have

\[ |q_1(x, y)| \leq C(w(B(x, 1)))^{-1} \exp(-cd(x, y)^{2\ell/(2\ell-1)}). \]
Theorem (J. Dziubański, A.H.)

Let \( \psi \) be a smooth radial function such that \( \text{supp} \psi \subseteq \{ \xi : \frac{1}{4} \leq \|\xi\| \leq 4 \} \) and \( \psi(\xi) \equiv 1 \) for \( \{ \xi : \frac{1}{2} \leq \|\xi\| \leq 2 \} \). If \( m \) satisfies

\[
M = \sup_{t > 0} \|\psi(\cdot)m(t\cdot)\|_{W^s_2} < \infty
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for some \( s > N \), then

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T_m f = \mathcal{F}^{-1}(m\mathcal{F}f),
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Idea of the proofs

Kernel estimate for $L$


2. Prove Gårding inequality and use theorem of Lions.

3. Pass to kernel pointwise estimate.

4. Perturbation of operator by Laplacian in order to use convolution with radial function technique.
Idea of the proofs

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**Kernel estimate for $L$**


2. Prove Gårding inequality and use theorem of Lions **no Leibniz rule!**
Idea of the proofs

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Hörmander’s multiplier theorem

1. Imitate the classical proof of Hörmander.
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Hörmander’s multiplier theorem

1. Imitate the classical proof of Hörmander.

2. Convolution with radial function technique.
Convolution with radial function technique

\[ L = (-1)^{\ell+1} \sum_{j=1}^{N} T_j^{2\ell}, \text{ associated multiplier } q(\xi) = \exp(- \sum_{j=1}^{N} |\xi_j|^{2\ell}). \]
Convolution with radial function technique

\[ L = (-1)^{\ell+1} \sum_{j=1}^{N} T_j^{2\ell}, \text{ associated multiplier } q(\xi) = \exp(-\sum_{j=1}^{N} |\xi_j|^{2\ell}). \]

Introduce \( L^{(\varepsilon)} = L - \varepsilon \Delta \), associated multiplier is

\[ q^{(\varepsilon)}(\xi) = \exp(-\sum_{j=1}^{N} |\xi_j|^{2\ell} + \varepsilon |\xi|^2). \]
Convolution with radial function technique

\( L = (-1)^{\ell+1} \sum_{j=1}^{N} T_j^{2\ell} \), associated multiplier \( q(\xi) = \exp\left( - \sum_{j=1}^{N} |\xi_j|^{2\ell} \right) \).

Introduce \( L^{(\varepsilon)} = L - \varepsilon \Delta \), associated multiplier is

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q^{(\varepsilon)}(\xi) = \exp\left( - \sum_{j=1}^{N} |\xi_j|^{2\ell} + \varepsilon |\xi|^2 \right).
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Then

\[
q(\xi) = q^{(\varepsilon)}(\xi) e^{-\varepsilon |\xi|^2},
\]
Convolution with radial function technique

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Introduce \( L(\varepsilon) = L - \varepsilon \Delta \), associated multiplier is

\[ q^{(\varepsilon)}(\xi) = \exp(-\sum_{j=1}^{N} |\xi_j|^{2\ell} + \varepsilon |\xi|^{2}). \]

Then

\[ q(\xi) = q^{(\varepsilon)}(\xi) e^{-\varepsilon |\xi|^2}, \]

so

\[ \tau_x(\mathcal{F}q)(-y) = \tau_x(\mathcal{F}q^{(\varepsilon)} * h_{\varepsilon})(-y) = (\mathcal{F}q^{(\varepsilon)}) * \tau_x(h_{\varepsilon})(-y). \]

Translation is on radial function!
Further problem

\( g \)-radial, \( f \) - not necessary radial
Further problem

g-radial, \( f \) - not necessary radial

**Classical case (trivial)**

\[
\| \tau_x (f \ast g) \|_{L^1(dw)} \leq \| f \|_{L^1(dw)} \| g \|_{L^\infty}
\]
Further problem

$g$-radial, $f$ - not necessary radial

**Classical case (trivial)**

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\| \tau_x (f \ast g) \|_{L^1(dw)} \leq \| f \|_{L^1(dw)} \| g \|_{L^\infty}
\]

**Dunkl case (no $L^1$-boundedness!)

\[
\| \tau_x (f \ast g) \|_{L^1(dw)} \leq \| f(\cdot)(1 + |\cdot|)^{N/2} \|_{L^1(dw)} \| g(\cdot)(1 + |\cdot|)^N \|_{L^\infty}
\]
Further problem

\( g \)-radial, \( f \) - not necessary radial

**Classical case (trivial)**

\[ \| \tau_x (f \ast g) \|_{L^1(dw)} \leq \| f \|_{L^1(dw)} \| g \|_{L^\infty} \]

**Dunkl case (no \( L^1 \)-boundedness!)**

\[ \| \tau_x (f \ast g) \|_{L^1(dw)} \leq \| f(\cdot)(1 + |\cdot|)^{N/2} \|_{L^1(dw)} \| g(\cdot)(1 + |\cdot|)^N \|_{L^\infty} \]

**Idea:** Pass to \( L^2 \) by Cauchy–Schwarz.

\[ \| \tau_x (f \ast g) \|_{L^1(dw)} \leq w(\text{supp } \tau_x (f \ast g))^{1/2} \| \tau_x (f \ast g) \|_{L^2(dw)} \]
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Question

Suppose that $f \in L^2(dw)$ is such that $\text{supp } f \subseteq B(0, 1)$. If we consider $f_x = f(x - \cdot)$, then $\text{supp } f_x \subseteq B(x, 1)$.

**Question:** What about $\text{supp } \tau_x f(\cdot)$?
Results of Amri, Anker and Sifi (Paley-Wiener approach) imply

$$\text{supp} \, \tau_x f(-\cdot) \subseteq \{y : \|x\| - 1 \leq \|y\| \leq \|x\| + 1\}.$$
Results of Rösler imply that if $f$ is \textit{radial}, then
\[
\text{supp } \tau_x f(- \cdot) \subseteq \mathcal{O}(B(x, 1)) = \bigcup_{g \in G} B(g(x), 1).
\]
Theorem (J. Dziubański, A.H.)

Let \( f \in L^2(dw) \), \( \text{supp} \ f \subseteq B(0, 1) \), and \( x \in \mathbb{R}^N \). Then

\[
\text{supp } \tau_x f (- \cdot) \subseteq O(B(x, 1)).
\]
What is the point?

The measure of $O(B(x, 1))$ is much smaller than the measure of 
\[ \{ y : \|x\| - 1 \leq \|y\| \leq \|x\| + 1 \}. \]
Let us denote

\[ g_L(x) = \max\{0, (1 - \|x\|^2)^L\}. \]
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\[ g_L(x) = \max\{0, (1 - \|x\|^2)\}^L. \]

For \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{N}_0^N = (\mathbb{N} \cup \{0\})^N \) we define
\[ T_0^\alpha = I, \quad T_1^\alpha := T_1^{\alpha_1} \circ T_2^{\alpha_2} \circ \ldots \circ T_N^{\alpha_N}. \]
Let us denote

$$g_L(x) = \max\{0, (1 - \|x\|^2)\}^L.$$  

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{N}_0^N = (\mathbb{N} \cup \{0\})^N$ we define

$$T_j^0 = I, \quad T^\alpha := T_1^{\alpha_1} \circ T_2^{\alpha_2} \circ \ldots \circ T_N^{\alpha_N}.$$  

1. By induction we show that if $p$ is a polynomial of degree $d$, then $p(x)g_L(x)$ can be written as

$$p(x)g_L(x) = \sum_{\ell=0}^d \sum_{\|\alpha\| \leq \ell} c_{\ell,\alpha} T^\alpha(g_{L+\ell})(x) \text{ for some } c_{\ell,\alpha} \in \mathbb{C}.$$
Let us denote
\[ g_L(x) = \max\{0, (1 - \|x\|^2)^L\}. \]

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\[ p(x)g_L(x) = \sum_{\ell=0}^d \sum_{\|\alpha\| \leq \ell} c_{\ell,\alpha} T^\alpha(g_L + \ell)(x) \]
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The key point is the fact that the Leibniz rule can be applied
\[ T_j(pg_L) = g_L(T_jp) + p(T_jg_L). \]
2 The set \( \{ p(\cdot)g_1(\cdot) : p \text{ is a polynomial} \} \) is dense in \( L^2(B(0, 1), dw) \).
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\( \tau_x \) is a contraction on \( L^2(dw) \), so for any \( \varepsilon > 0 \) there is a polynomial \( p \) such that

\[
\| \tau_x f - \tau_x (pg_1) \|_{L^2(dw)} \leq \| f - pg_1 \|_{L^2(dw)} < \varepsilon. \tag{*}
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4 The Dunkl translations commute with the Dunkl operators, so

\[
\tau_x(pg_1)(-y) = \sum_{\ell=0}^{d} \sum_{\|\alpha\| \leq \ell} c_{\ell,\alpha} T^\alpha \tau_x(g_{1+\ell})(-y).
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5. By the results of Rösler \( \text{supp} \ T^\alpha \tau_x(g_1+\ell) \subseteq O(B(x,1)) \), so \( (\star) \) implies the claim.
Thank you for your attention.


J. Dziubański and A. Hejna, *On semigroups generated by sums of even powers of Dunkl operators*, arxiv.