The good, the bad and the random cubes

Tuomas Hytönen[1]

University of Helsinki, Finland

Probability and Analysis,
20–24 May 2019, Bedlewo, Poland

[1]Member of the Centre of Excellence in Analysis and Dynamics Research.

Calderón–Zygmund decomposition 1952

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"
- $\mathscr{D} = \{2^{-k}([0,1)^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"
- $\mathscr{D} = \{2^{-k}([0,1)^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$
- $f \in L^1(\mathbb{R}^d)$, $\lambda > 0$, to prove: $|\{|Tf| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_1$.

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"
- $\mathscr{D} = \{2^{-k}([0,1)^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$
- $f \in L^1(\mathbb{R}^d)$, $\lambda > 0$, to prove: $|\{|Tf| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_1$.
- Good $Q \in \mathscr{D}$: $\langle |f| \rangle_Q \leqslant \lambda$.

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"
- $\mathscr{D} = \{2^{-k}([0,1)^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$
- $f \in L^1(\mathbb{R}^d)$, $\lambda > 0$, to prove: $|\{|Tf| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_1$.
- Good $Q \in \mathscr{D}$: $\langle |f| \rangle_Q \leqslant \lambda$.
- Bad $Q \in \mathscr{D}$: $\langle |f| \rangle_Q > \lambda$.

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"
- $\mathscr{D} = \{2^{-k}([0,1)^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$
- $f \in L^1(\mathbb{R}^d)$, $\lambda > 0$, to prove: $|\{|Tf| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_1$.
- Good $Q \in \mathscr{D}$: $\langle |f| \rangle_Q \leqslant \lambda$.
- Bad $Q \in \mathscr{D}$: $\langle |f| \rangle_Q > \lambda$.
- Maximal bad $Q_i \in \mathscr{D}$: $\langle |f| \rangle_{Q_i} \leqslant 2^d \langle |f| \rangle_{\hat{Q}_i} \leqslant 2^d \lambda$.

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"
- $\mathscr{D} = \{2^{-k}([0,1)^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$
- $f \in L^1(\mathbb{R}^d)$, $\lambda > 0$, to prove: $|\{|Tf| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_1$.
- Good $Q \in \mathscr{D}$: $\langle |f| \rangle_Q \leqslant \lambda$.
- Bad $Q \in \mathscr{D}$: $\langle |f| \rangle_Q > \lambda$.
- Maximal bad $Q_i \in \mathscr{D}$: $\langle |f| \rangle_{Q_i} \leqslant 2^d \langle |f| \rangle_{\hat{Q}_i} \leqslant 2^d \lambda$.
- Good $g = f \cdot 1_{(\bigcup Q_i)^c} + \sum_i \langle f \rangle_{Q_i} \cdot 1_{Q_i}$.

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"
- $\mathscr{D} = \{2^{-k}([0,1)^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$
- $f \in L^1(\mathbb{R}^d)$, $\lambda > 0$, to prove: $|\{|Tf| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_1$.
- Good $Q \in \mathscr{D}$: $\langle |f| \rangle_Q \leqslant \lambda$.
- Bad $Q \in \mathscr{D}$: $\langle |f| \rangle_Q > \lambda$.
- Maximal bad $Q_i \in \mathscr{D}$: $\langle |f| \rangle_{Q_i} \leqslant 2^d \langle |f| \rangle_{\hat{Q}_i} \leqslant 2^d \lambda$.
- Good $g = f \cdot 1_{(\bigcup Q_i)^c} + \sum_i \langle f \rangle_{Q_i} \cdot 1_{Q_i}$.
- $\|g\|_1 \leqslant \|f\|_1$, $\|g\|_\infty \leqslant 2^d\lambda$, $\|g\|_2^2 \leqslant 2^d\lambda\|f\|_1$: controlled size.

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"
- $\mathscr{D} = \{2^{-k}([0,1)^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$
- $f \in L^1(\mathbb{R}^d)$, $\lambda > 0$, to prove: $|\{|Tf| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_1$.
- Good $Q \in \mathscr{D}$: $\langle |f| \rangle_Q \leqslant \lambda$.
- Bad $Q \in \mathscr{D}$: $\langle |f| \rangle_Q > \lambda$.
- Maximal bad $Q_i \in \mathscr{D}$: $\langle |f| \rangle_{Q_i} \leqslant 2^d \langle |f| \rangle_{\hat{Q}_i} \leqslant 2^d \lambda$.
- Good $g = f \cdot 1_{(\bigcup Q_i)^c} + \sum_i \langle f \rangle_{Q_i} \cdot 1_{Q_i}$.
- $\|g\|_1 \leqslant \|f\|_1$, $\|g\|_\infty \leqslant 2^d\lambda$, $\|g\|_2^2 \leqslant 2^d \lambda \|f\|_1$: controlled size.
- T is L^2-bounded $\Rightarrow |\{|Tg| > \lambda/2\}| \leqslant C\lambda^{-2}\|g\|_2^2 \leqslant C\lambda^{-1}\|f\|_1$.

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"
- $\mathscr{D} = \{2^{-k}([0,1)^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$
- $f \in L^1(\mathbb{R}^d)$, $\lambda > 0$, to prove: $|\{|Tf| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_1$.
- Good $Q \in \mathscr{D}$: $\langle |f| \rangle_Q \leqslant \lambda$.
- Bad $Q \in \mathscr{D}$: $\langle |f| \rangle_Q > \lambda$.
- Maximal bad $Q_i \in \mathscr{D}$: $\langle |f| \rangle_{Q_i} \leqslant 2^d \langle |f| \rangle_{\hat{Q}_i} \leqslant 2^d \lambda$.
- Good $g = f \cdot 1_{(\bigcup Q_i)^c} + \sum_i \langle f \rangle_{Q_i} \cdot 1_{Q_i}$.
- $\|g\|_1 \leqslant \|f\|_1$, $\|g\|_\infty \leqslant 2^d \lambda$, $\|g\|_2^2 \leqslant 2^d \lambda \|f\|_1$: controlled size.
- T is L^2-bounded $\Rightarrow |\{|Tg| > \lambda/2\}| \leqslant C\lambda^{-2}\|g\|_2^2 \leqslant C\lambda^{-1}\|f\|_1$.
- Bad $b = f - g = \sum_i (f - \langle f \rangle_{Q_i}) \cdot 1_{Q_i}$: support + cancellation.

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"
- $\mathscr{D} = \{2^{-k}([0,1)^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$
- $f \in L^1(\mathbb{R}^d)$, $\lambda > 0$, to prove: $|\{|Tf| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_1$.
- Good $Q \in \mathscr{D}$: $\langle |f| \rangle_Q \leqslant \lambda$.
- Bad $Q \in \mathscr{D}$: $\langle |f| \rangle_Q > \lambda$.
- Maximal bad $Q_i \in \mathscr{D}$: $\langle |f| \rangle_{Q_i} \leqslant 2^d \langle |f| \rangle_{\hat{Q}_i} \leqslant 2^d\lambda$.
- Good $g = f \cdot 1_{(\bigcup Q_i)^c} + \sum_i \langle f \rangle_{Q_i} \cdot 1_{Q_i}$.
- $\|g\|_1 \leqslant \|f\|_1$, $\|g\|_\infty \leqslant 2^d\lambda$, $\|g\|_2^2 \leqslant 2^d\lambda\|f\|_1$: controlled size.
- T is L^2-bounded $\Rightarrow |\{|Tg| > \lambda/2\}| \leqslant C\lambda^{-2}\|g\|_2^2 \leqslant C\lambda^{-1}\|f\|_1$.
- Bad $b = f - g = \sum_i (f - \langle f \rangle_{Q_i}) \cdot 1_{Q_i}$: support + cancellation.
- $Tb(x) = \int K(x,y)b(y)\,\mathrm{d}y = \sum_i \int_{Q_i} [K(x,y) - K(x,z_i)]b_i(y)\,\mathrm{d}y$.

Calderón–Zygmund decomposition 1952

- "In the 1D case this lemma is contained in a lemma by F. Riesz"
- $\mathscr{D} = \{2^{-k}([0,1)^d + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$
- $f \in L^1(\mathbb{R}^d)$, $\lambda > 0$, to prove: $|\{|Tf| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_1$.
- Good $Q \in \mathscr{D}$: $\langle |f| \rangle_Q \leqslant \lambda$.
- Bad $Q \in \mathscr{D}$: $\langle |f| \rangle_Q > \lambda$.
- Maximal bad $Q_i \in \mathscr{D}$: $\langle |f| \rangle_{Q_i} \leqslant 2^d \langle |f| \rangle_{\hat{Q}_i} \leqslant 2^d \lambda$.
- Good $g = f \cdot 1_{(\bigcup Q_i)^c} + \sum_i \langle f \rangle_{Q_i} \cdot 1_{Q_i}$.
- $\|g\|_1 \leqslant \|f\|_1$, $\|g\|_\infty \leqslant 2^d\lambda$, $\|g\|_2^2 \leqslant 2^d\lambda\|f\|_1$: controlled size.
- T is L^2-bounded $\Rightarrow |\{|Tg| > \lambda/2\}| \leqslant C\lambda^{-2}\|g\|_2^2 \leqslant C\lambda^{-1}\|f\|_1$.
- Bad $b = f - g = \sum_i (f - \langle f \rangle_{Q_i}) \cdot 1_{Q_i}$: support + cancellation.
- $Tb(x) = \int K(x,y)b(y)\,\mathrm{d}y = \sum_i \int_{Q_i} [K(x,y) - K(x,z_i)]b_i(y)\,\mathrm{d}y$.
- $|\{|Tb| > \lambda/2\}| \leqslant \ldots$ (kernel bounds etc.)$\ldots \leqslant C\lambda^{-1}\|f\|_1$.

Weighted version 1973–2009–2017

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973:

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w)$

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log?

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973:
  $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log? Nazarov et al: At least $\log^{\varepsilon}$,

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log? Nazarov et al: At least $\log^{\varepsilon}$, $\varepsilon = \frac{1}{7}$,

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log? Nazarov et al: At least $\log^{\varepsilon}$, $\varepsilon = \frac{1}{7}, \frac{1}{5},$

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log? Nazarov et al: At least $\log^{\varepsilon}$, $\varepsilon = \frac{1}{7}, \frac{1}{5}, \frac{1}{3}$,

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log? Nazarov et al: At least $\log^{\varepsilon}$, $\varepsilon = \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \dots$?

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log? Nazarov et al: At least $\log^{\varepsilon}$, $\varepsilon = \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \ldots$?
- Need $\varepsilon = 1$ confirmed by Lerner, Nazarov, Ombrosi 2017a.

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log? Nazarov et al: At least $\log^{\varepsilon}$, $\varepsilon = \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \ldots$?
- Need $\varepsilon = 1$ confirmed by Lerner, Nazarov, Ombrosi 2017a.
- Proof of $(*)$ by CZ decomposition.

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log? Nazarov et al: At least $\log^{\varepsilon}$, $\varepsilon = \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \ldots$?
- Need $\varepsilon = 1$ confirmed by Lerner, Nazarov, Ombrosi 2017a.
- Proof of $(*)$ by CZ decomposition.
- $w(\{|Tb| > \lambda/2\}) \leqslant C\lambda^{-1}[w]_{A_1}\|f\|_{L^1(w)}$ ($\sim$ straightforward)

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log? Nazarov et al: At least $\log^\varepsilon$, $\varepsilon = \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \ldots$?
- Need $\varepsilon = 1$ confirmed by Lerner, Nazarov, Ombrosi 2017a.
- Proof of $(*)$ by CZ decomposition.
- $w(\{|Tb| > \lambda/2\}) \leqslant C\lambda^{-1}[w]_{A_1}\|f\|_{L^1(w)}$ ($\sim$ straightforward)
- g: tricky, uses precise L^p bound for $p = 1 + \varepsilon$ etc., still gives log!

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log? Nazarov et al: At least $\log^\varepsilon$, $\varepsilon = \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \ldots$?
- Need $\varepsilon = 1$ confirmed by Lerner, Nazarov, Ombrosi 2017a.
- Proof of $(*)$ by CZ decomposition.
- $w(\{|Tb| > \lambda/2\}) \leqslant C\lambda^{-1}[w]_{A_1}\|f\|_{L^1(w)}$ ($\sim$ straightforward)
- g: tricky, uses precise L^p bound for $p = 1 + \varepsilon$ etc., still gives log!
- "The bad part is good, but the good part is bad." (C. Pérez)

Weighted version 1973–2009–2017

- Muckenhoupt class A_1: $Mw(x) \leqslant Cw(x)$; $[w]_{A_1} :=$ best C.
- Muckenhoupt 1972; Hunt, Muckenhoupt, Wheeden 1973: $w \in A_1 \Leftrightarrow M : L^1(w) \to L^{1,\infty}(w) \Leftrightarrow H : L^1(w) \to L^{1,\infty}(w)$.
- Sharp bound ("A_1 theorem"): Lerner, Ombrosi, Pérez 2009:

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C}{\lambda}[w]_{A_1}(1 + \log[w]_{A_1})\|f\|_{L^1(w)}. \qquad (*)$$

- Need log? Nazarov et al: At least $\log^\varepsilon$, $\varepsilon = \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \ldots$?
- Need $\varepsilon = 1$ confirmed by Lerner, Nazarov, Ombrosi 2017a.
- Proof of $(*)$ by CZ decomposition.
- $w(\{|Tb| > \lambda/2\}) \leqslant C\lambda^{-1}[w]_{A_1}\|f\|_{L^1(w)}$ ($\sim$ straightforward)
- g: tricky, uses precise L^p bound for $p = 1 + \varepsilon$ etc., still gives log!
- "The bad part is good, but the good part is bad." (C. Pérez)

Non-homogeneous singular integrals $\sim$ 2000–

Non-homogeneous singular integrals $\sim$ 2000–

- $Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,\mathrm{d}\mu(y), \quad \mu(B(x,r)) \leqslant Cr^n,\ n \in (0,d].$

Non-homogeneous singular integrals $\sim$ 2000–

- $Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) \, \mathrm{d}\mu(y)$, $\quad \mu(B(x,r)) \leqslant Cr^n$, $n \in (0,d]$.
- $|K(x,y)| \leqslant C|x-y|^{-n}$, Hölder-continuity bounds.

Non-homogeneous singular integrals $\sim$ 2000–

- $Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,\mathrm{d}\mu(y), \quad \mu(B(x,r)) \leqslant Cr^n,\ n \in (0,d]$.
- $|K(x,y)| \leqslant C|x-y|^{-n}$, Hölder-continuity bounds.
- No lower bound $\mu(B(x,r)) \geqslant \ldots$ assumed!

Non-homogeneous singular integrals $\sim$ 2000–

- $Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) \, \mathrm{d}\mu(y)$, $\quad \mu(B(x,r)) \leqslant Cr^n$, $n \in (0,d]$.
- $|K(x,y)| \leqslant C|x-y|^{-n}$, Hölder-continuity bounds.
- No lower bound $\mu(B(x,r)) \geqslant \ldots$ assumed!
- Key case: $d = 2$, $n = 1$, $K(x,y) = (x-y)^{-1}$: Cauchy integral!

Non-homogeneous singular integrals $\sim$ 2000–

- $Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) \,\mathrm{d}\mu(y), \quad \mu(B(x,r)) \leqslant Cr^n,\ n \in (0,d]$.
- $|K(x,y)| \leqslant C|x-y|^{-n}$, Hölder-continuity bounds.
- No lower bound $\mu(B(x,r)) \geqslant \ldots$ assumed!
- Key case: $d=2,\ n=1,\ K(x,y) = (x-y)^{-1}$: Cauchy integral!
- $\mathscr{D}$ still there $\Rightarrow$ dyadic martingales w.r.t. measure μ.

Non-homogeneous singular integrals $\sim$ 2000–

- $Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\,\mathrm{d}\mu(y)$, $\quad \mu(B(x,r)) \leqslant Cr^n$, $n \in (0,d]$.
- $|K(x,y)| \leqslant C|x-y|^{-n}$, Hölder-continuity bounds.
- No lower bound $\mu(B(x,r)) \geqslant \ldots$ assumed!
- Key case: $d = 2$, $n = 1$, $K(x,y) = (x-y)^{-1}$: Cauchy integral!
- $\mathscr{D}$ still there $\Rightarrow$ dyadic martingales w.r.t. measure μ.
- $f = \sum_{Q \in \mathscr{D}} \Delta_Q f$

Non-homogeneous singular integrals $\sim$ 2000–

- $Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,\mathrm{d}\mu(y)$, $\quad \mu(B(x,r)) \leqslant Cr^n$, $n \in (0,d]$.
- $|K(x,y)| \leqslant C|x-y|^{-n}$, $\qquad$ Hölder-continuity bounds.
- No lower bound $\mu(B(x,r)) \geqslant \ldots$ assumed!
- Key case: $d = 2$, $n = 1$, $K(x,y) = (x-y)^{-1}$: Cauchy integral!
- $\mathscr{D}$ still there $\Rightarrow$ dyadic martingales w.r.t. measure μ.
- $f = \sum_{Q \in \mathscr{D}} \Delta_Q f$
- $\langle Tf, g \rangle = \sum_{Q,R \in \mathscr{D}} \langle T\Delta_Q f, \Delta_R g \rangle$.

Non-homogeneous singular integrals $\sim$ 2000–

- $Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,\mathrm{d}\mu(y), \quad \mu(B(x,r)) \leqslant Cr^n,\ n \in (0,d]$.
- $|K(x,y)| \leqslant C|x-y|^{-n}$, Hölder-continuity bounds.
- No lower bound $\mu(B(x,r)) \geqslant \ldots$ assumed!
- Key case: $d=2$, $n=1$, $K(x,y) = (x-y)^{-1}$: Cauchy integral!
- $\mathscr{D}$ still there $\Rightarrow$ dyadic martingales w.r.t. measure μ.
- $f = \sum_{Q\in\mathscr{D}} \Delta_Q f$
- $\langle Tf, g\rangle = \sum_{Q,R\in\mathscr{D}} \langle T\Delta_Q f, \Delta_R g\rangle$.
- Many pairings $\sim$ classical, but some give trouble.

Non-homogeneous singular integrals $\sim$ 2000–

- $Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\,\mathrm{d}\mu(y)$, $\quad \mu(B(x,r)) \leqslant Cr^n$, $n \in (0,d]$.
- $|K(x,y)| \leqslant C|x-y|^{-n}$, $\qquad$ Hölder-continuity bounds.
- No lower bound $\mu(B(x,r)) \geqslant \ldots$ assumed!
- Key case: $d = 2$, $n = 1$, $K(x,y) = (x-y)^{-1}$: Cauchy integral!
- $\mathscr{D}$ still there $\Rightarrow$ dyadic martingales w.r.t. measure μ.
- $f = \sum_{Q \in \mathscr{D}} \Delta_Q f$
- $\langle Tf, g\rangle = \sum_{Q,R \in \mathscr{D}} \langle T\Delta_Q f, \Delta_R g\rangle$.
- Many pairings $\sim$ classical, but some give trouble.
- Problematic: small Q close to ∂R with $\ell(R) \gg \ell(Q)$.

Non-homogeneous singular integrals $\sim$ 2000–

- $Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,\mathrm{d}\mu(y), \quad \mu(B(x,r)) \leqslant Cr^n,\ n \in (0,d]$.
- $|K(x,y)| \leqslant C|x-y|^{-n}$, Hölder-continuity bounds.
- No lower bound $\mu(B(x,r)) \geqslant \ldots$ assumed!
- Key case: $d = 2$, $n = 1$, $K(x,y) = (x-y)^{-1}$: Cauchy integral!
- $\mathscr{D}$ still there $\Rightarrow$ dyadic martingales w.r.t. measure μ.
- $f = \sum_{Q\in\mathscr{D}} \Delta_Q f$
- $\langle Tf, g\rangle = \sum_{Q,R\in\mathscr{D}} \langle T\Delta_Q f, \Delta_R g\rangle$.
- Many pairings $\sim$ classical, but some give trouble.
- Problematic: small Q close to ∂R with $\ell(R) \gg \ell(Q)$.
- These should be "rare".

Random dyadic cubes after Nazarov, Treil, Volberg

Random dyadic cubes after Nazarov, Treil, Volberg

- Parameter $\beta = (\beta_j)_{j \in \mathbb{Z}} \in (\{0, 1\}^d)^{\mathbb{Z}} = \Omega$.

Random dyadic cubes after Nazarov, Treil, Volberg

- Parameter $\beta = (\beta_j)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}} = \Omega$.
- $Q\dot{+}\beta := Q + \sum_{j:2^j<\ell(Q)} \beta_j 2^j$.

Random dyadic cubes after Nazarov, Treil, Volberg

- Parameter $\beta = (\beta_j)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}} = \Omega$.
- $Q\dot{+}\beta := Q + \sum_{j:2^j<\ell(Q)} \beta_j 2^j$.
- $\mathscr{D}^\beta := \{Q\dot{+}\beta : Q \in \mathscr{D}\}$ another "dyadic system".

Random dyadic cubes after Nazarov, Treil, Volberg

- Parameter $\beta = (\beta_j)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}} = \Omega$.
- $Q\dot{+}\beta := Q + \sum_{j:2^j<\ell(Q)} \beta_j 2^j$.
- $\mathscr{D}^\beta := \{Q\dot{+}\beta : Q \in \mathscr{D}\}$ another "dyadic system".
- $f = \sum_{Q\in\mathscr{D}} \Delta_Q f = \sum_{P\in\mathscr{D}^\beta} \Delta_P f \quad \forall\beta\in\Omega$ (no "preferred choice")

Random dyadic cubes after Nazarov, Treil, Volberg

- Parameter $\beta = (\beta_j)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}} = \Omega$.
- $Q\dot{+}\beta := Q + \sum_{j:2^j<\ell(Q)} \beta_j 2^j$.
- $\mathscr{D}^\beta := \{Q\dot{+}\beta : Q \in \mathscr{D}\}$ another "dyadic system".
- $f = \sum_{Q\in\mathscr{D}} \Delta_Q f = \sum_{P\in\mathscr{D}^\beta} \Delta_P f \quad \forall\beta \in \Omega$ (no "preferred choice")
- $\{\mathscr{D}^\beta : \beta \in \Omega\}$ parametrised family of dyadic systems.

Random dyadic cubes after Nazarov, Treil, Volberg

- Parameter $\beta = (\beta_j)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}} = \Omega$.
- $Q \dot{+} \beta := Q + \sum_{j:2^j<\ell(Q)} \beta_j 2^j$.
- $\mathscr{D}^\beta := \{Q\dot{+}\beta : Q \in \mathscr{D}\}$ another "dyadic system".
- $f = \sum_{Q\in\mathscr{D}} \Delta_Q f = \sum_{P\in\mathscr{D}^\beta} \Delta_P f \quad \forall\beta\in\Omega$ (no "preferred choice")
- $\{\mathscr{D}^\beta : \beta\in\Omega\}$ parametrised family of dyadic systems.
- Equip Ω with (the natural) probability $\Rightarrow$ random dyadic cubes.

Bad random cubes

Bad random cubes

- P close to ∂S with $\ell(S) \gg \ell(P)$ is "bad".

Bad random cubes

- P close to ∂S with $\ell(S) \gg \ell(P)$ is “bad”.
- Quantify: $P \in \mathscr{D}^\beta$ is (r,γ)-bad if:
  $\exists S \in \mathscr{D}^\beta$: $\ell(S) \geqslant 2^r \ell(P)$ & $\mathrm{dist}(P, \partial S) \leqslant \ell(P)^\gamma \ell(S)^{1-\gamma}$.

Bad random cubes

- P close to ∂S with $\ell(S) \gg \ell(P)$ is "bad".
- Quantify: $P \in \mathscr{D}^\beta$ is (r, γ)-bad if:
  $\exists S \in \mathscr{D}^\beta$: $\ell(S) \geqslant 2^r \ell(P)$ & $\operatorname{dist}(P, \partial S) \leqslant \ell(P)^\gamma \ell(S)^{1-\gamma}$.
- W.l.o.g. $S = P^{(k)} :=$ kth generation ancestor of P with $k \geqslant r$.

Bad random cubes

- P close to ∂S with $\ell(S) \gg \ell(P)$ is "bad".
- Quantify: $P \in \mathscr{D}^\beta$ is (r,γ)-bad if:
  $\exists S \in \mathscr{D}^\beta$: $\ell(S) \geqslant 2^r \ell(P)$ & $\operatorname{dist}(P, \partial S) \leqslant \ell(P)^\gamma \ell(S)^{1-\gamma}$.
- W.l.o.g. $S = P^{(k)} :=$ kth generation ancestor of P with $k \geqslant r$.
- $P = Q \dot{+} \beta = Q + \sum_{j:2^j < \ell(Q)} \beta_j 2^j$, $Q \in \mathscr{D}^0$.

Bad random cubes

- P close to ∂S with $\ell(S) \gg \ell(P)$ is "bad".
- Quantify: $P \in \mathscr{D}^\beta$ is (r,γ)-bad if:
  $\exists S \in \mathscr{D}^\beta$: $\ell(S) \geqslant 2^r \ell(P)$ & $\mathrm{dist}(P, \partial S) \leqslant \ell(P)^\gamma \ell(S)^{1-\gamma}$.
- W.l.o.g. $S = P^{(k)}$:= kth generation ancestor of P with $k \geqslant r$.
- $P = Q \dot{+} \beta = Q + \sum_{j:2^j<\ell(Q)} \beta_j 2^j$, $Q \in \mathscr{D}^0$.
- $S = R + \sum_{j:2^j<\ell(R)} \beta_j 2^j = R + \sum_{j:2^j<\ell(Q)} \beta_j 2^j + \sum_{j:\ell(Q)\leqslant 2^j<\ell(R)} \beta_j 2^j$.

Bad random cubes

- P close to ∂S with $\ell(S) \gg \ell(P)$ is "bad".
- Quantify: $P \in \mathscr{D}^\beta$ is (r,γ)-bad if:
  $\exists S \in \mathscr{D}^\beta$: $\ell(S) \geqslant 2^r \ell(P)$ & $\operatorname{dist}(P, \partial S) \leqslant \ell(P)^\gamma \ell(S)^{1-\gamma}$.
- W.l.o.g. $S = P^{(k)} :=$ kth generation ancestor of P with $k \geqslant r$.
- $P = Q \dot{+} \beta = Q + \sum_{j:2^j<\ell(Q)} \beta_j 2^j$, $Q \in \mathscr{D}^0$.
- $S = R + \sum_{j:2^j<\ell(R)} \beta_j 2^j = R + \sum_{j:2^j<\ell(Q)} \beta_j 2^j + \sum_{j:\ell(Q)\leqslant 2^j<\ell(R)} \beta_j 2^j$.
- Position of P and relative position of P in S are independent!

Bad random cubes

- P close to ∂S with $\ell(S) \gg \ell(P)$ is "bad".
- Quantify: $P \in \mathscr{D}^\beta$ is (r,γ)-bad if:
  $\exists S \in \mathscr{D}^\beta$: $\ell(S) \geqslant 2^r \ell(P)$ & $\mathrm{dist}(P, \partial S) \leqslant \ell(P)^\gamma \ell(S)^{1-\gamma}$.
- W.l.o.g. $S = P^{(k)} :=$ kth generation ancestor of P with $k \geqslant r$.
- $P = Q \dot{+} \beta = Q + \sum_{j:2^j<\ell(Q)} \beta_j 2^j$, $Q \in \mathscr{D}^0$.
- $S = R + \sum_{j:2^j<\ell(R)} \beta_j 2^j = R + \sum_{j:2^j<\ell(Q)} \beta_j 2^j + \sum_{j:\ell(Q)\leqslant 2^j<\ell(R)} \beta_j 2^j$.
- Position of P and relative position of P in S are independent!
- Bad cubes in $\delta = \ell(P)^\gamma \ell(S)^{1-\gamma}$-nbhd of ∂S, their portion of S is $\sim \delta \ell(S)^{d-1}/|S| = (\ell(P)/\ell(S))^\gamma = 2^{-k\gamma}$.

Bad random cubes

- P close to ∂S with $\ell(S) \gg \ell(P)$ is "bad".
- Quantify: $P \in \mathscr{D}^\beta$ is (r,γ)-bad if:
  $\exists S \in \mathscr{D}^\beta$: $\ell(S) \geqslant 2^r \ell(P)$ & $\mathrm{dist}(P, \partial S) \leqslant \ell(P)^\gamma \ell(S)^{1-\gamma}$.
- W.l.o.g. $S = P^{(k)} :=$ kth generation ancestor of P with $k \geqslant r$.
- $P = Q \dot{+} \beta = Q + \sum_{j:2^j<\ell(Q)} \beta_j 2^j$, $Q \in \mathscr{D}^0$.
- $S = R + \sum_{j:2^j<\ell(R)} \beta_j 2^j = R + \sum_{j:2^j<\ell(Q)} \beta_j 2^j + \sum_{j:\ell(Q)\leqslant 2^j<\ell(R)} \beta_j 2^j$.
- Position of P and relative position of P in S are independent!
- Bad cubes in $\delta = \ell(P)^\gamma \ell(S)^{1-\gamma}$-nbhd of ∂S, their portion of S is $\sim \delta \ell(S)^{d-1}/|S| = (\ell(P)/\ell(S))^\gamma = 2^{-k\gamma}$.
- $\mathbb{P}(Q \dot{+} \beta \text{ is bad}) \lesssim \sum_{k=r}^\infty 2^{-k\gamma} \lesssim 2^{-r\gamma} < \eta$ for $r \geqslant r(d,\gamma,\eta)$.

Bad random functions

Bad random functions

- $f = \sum_{Q \in \mathscr{D}} \Delta_{Q \dot{+} \beta} f = f_{\text{good}} + f_{\text{bad}}$

Bad random functions

- $f = \sum_{Q \in \mathscr{D}} \Delta_{Q \dot{+} \beta} f = f_{\text{good}} + f_{\text{bad}}$
- $f_{\text{bad}} = \sum_{Q \in \mathscr{D}} 1_{\text{bad}}(Q \dot{+} \beta) \Delta_{Q \dot{+} \beta} f.$

Bad random functions

- $f = \sum_{Q \in \mathscr{D}} \Delta_{Q \dot{+} \beta} f = f_{\text{good}} + f_{\text{bad}}$
- $f_{\text{bad}} = \sum_{Q \in \mathscr{D}} 1_{\text{bad}}(Q \dot{+} \beta) \Delta_{Q \dot{+} \beta} f$.
- $\|f\|^2_{L^2(\mu)} = \sum_{Q \in \mathscr{D}} \|\Delta_{Q \dot{+} \beta} f\|^2_{L^2(\mu)}$

Bad random functions

- $f = \sum_{Q\in\mathscr{D}} \Delta_{Q\dot{+}\beta} f = f_{\text{good}} + f_{\text{bad}}$
- $f_{\text{bad}} = \sum_{Q\in\mathscr{D}} 1_{\text{bad}}(Q\dot{+}\beta)\Delta_{Q\dot{+}\beta} f$.
- $\|f\|^2_{L^2(\mu)} = \sum_{Q\in\mathscr{D}} \|\Delta_{Q\dot{+}\beta} f\|^2_{L^2(\mu)}$

$$\begin{aligned}
\mathbb{E}\|f_{\text{bad}}\|^2_{L^2(\mu)} &= \sum_{Q\in\mathscr{D}} \mathbb{E}\left(1_{\text{bad}}(Q\dot{+}\beta)\|\Delta_{Q\dot{+}\beta} f\|^2_{L^2(\mu)}\right) \\
&= \sum_{Q\in\mathscr{D}} \mathbb{E}\left(1_{\text{bad}}(Q\dot{+}\beta)\right)\mathbb{E}\left(\|\Delta_{Q\dot{+}\beta} f\|^2_{L^2(\mu)}\right) \quad \text{indep!} \\
&\leqslant \sum_{Q\in\mathscr{D}} \eta\mathbb{E}\left(\|\Delta_{Q\dot{+}\beta} f\|^2_{L^2(\mu)}\right) = \eta\|f\|^2_{L^2(\mu)}.
\end{aligned}$$

Operator estimates via good/bad decomposition

Operator estimates via good/bad decomposition

- Suppose we can prove $|\langle Tf_{\text{good}}, g_{\text{good}}\rangle| \leqslant C_{\text{good}} \|f_{\text{good}}\|_2 \|g_{\text{good}}\|_2$.

Operator estimates via good/bad decomposition

- Suppose we can prove $|\langle Tf_{\text{good}}, g_{\text{good}}\rangle| \leqslant C_{\text{good}}\|f_{\text{good}}\|_2\|g_{\text{good}}\|_2$.
- (Actually very highly nontrivial, but doable!)

Operator estimates via good/bad decomposition

- Suppose we can prove $|\langle Tf_{\text{good}}, g_{\text{good}} \rangle| \leqslant C_{\text{good}} \|f_{\text{good}}\|_2 \|g_{\text{good}}\|_2$.
- (Actually very highly nontrivial, but doable!)
- Bad part: "It could be difficult unless it was easy." (A. McIntosh)

Operator estimates via good/bad decomposition

- Suppose we can prove $|\langle Tf_{\text{good}}, g_{\text{good}}\rangle| \leqslant C_{\text{good}}\|f_{\text{good}}\|_2\|g_{\text{good}}\|_2$.
- (Actually very highly nontrivial, but doable!)
- Bad part: "It could be difficult unless it was easy." (A. McIntosh)

$$\begin{aligned}
|\langle Tf, g\rangle| &= \mathbb{E}|\langle Tf_{\text{good}}, g_{\text{good}}\rangle + \langle Tf_{\text{bad}}, g_{\text{good}}\rangle + \langle Tf, g_{\text{bad}}\rangle| \\
&\leqslant C_{\text{good}}\|f\|_2\|g\|_2 + \|T\|\mathbb{E}\|f_{\text{bad}}\|_2\|g\|_2 + \|T\|\|f\|_2\mathbb{E}\|g_{\text{bad}}\|_2 \\
&\leqslant \left(C_{\text{good}} + 2\sqrt{\eta}\|T\|\right)\|f\|_2\|g\|_2
\end{aligned}$$

Operator estimates via good/bad decomposition

- Suppose we can prove $|\langle Tf_{\text{good}}, g_{\text{good}}\rangle| \leqslant C_{\text{good}}\|f_{\text{good}}\|_2\|g_{\text{good}}\|_2$.
- (Actually very highly nontrivial, but doable!)
- Bad part: "It could be difficult unless it was easy." (A. McIntosh)

$$\begin{aligned}
|\langle Tf, g\rangle| &= \mathbb{E}|\langle Tf_{\text{good}}, g_{\text{good}}\rangle + \langle Tf_{\text{bad}}, g_{\text{good}}\rangle + \langle Tf, g_{\text{bad}}\rangle| \\
&\leqslant C_{\text{good}}\|f\|_2\|g\|_2 + \|T\|\mathbb{E}\|f_{\text{bad}}\|_2\|g\|_2 + \|T\|\|f\|_2\mathbb{E}\|g_{\text{bad}}\|_2 \\
&\leqslant \big(C_{\text{good}} + 2\sqrt{\eta}\|T\|\big)\|f\|_2\|g\|_2
\end{aligned}$$

- $\Rightarrow \|T\| \leqslant C_{\text{good}} + 2\sqrt{\eta}\|T\| \Rightarrow \|T\| \leqslant C_{\text{good}}(1-2\sqrt{\eta})^{-1}$.

Operator estimates via good/bad decomposition

- Suppose we can prove $|\langle Tf_{\text{good}}, g_{\text{good}}\rangle| \leqslant C_{\text{good}}\|f_{\text{good}}\|_2\|g_{\text{good}}\|_2$.
- (Actually very highly nontrivial, but doable!)
- Bad part: "It could be difficult unless it was easy." (A. McIntosh)

$$\begin{aligned} |\langle Tf, g\rangle| &= \mathbb{E}|\langle Tf_{\text{good}}, g_{\text{good}}\rangle + \langle Tf_{\text{bad}}, g_{\text{good}}\rangle + \langle Tf, g_{\text{bad}}\rangle| \\ &\leqslant C_{\text{good}}\|f\|_2\|g\|_2 + \|T\|\mathbb{E}\|f_{\text{bad}}\|_2\|g\|_2 + \|T\|\|f\|_2\mathbb{E}\|g_{\text{bad}}\|_2 \\ &\leqslant \big(C_{\text{good}} + 2\sqrt{\eta}\|T\|\big)\|f\|_2\|g\|_2 \end{aligned}$$

- $\Rightarrow \|T\| \leqslant C_{\text{good}} + 2\sqrt{\eta}\|T\| \Rightarrow \|T\| \leqslant C_{\text{good}}(1-2\sqrt{\eta})^{-1}$.
- $\Rightarrow$ "$T1$ theorem" (= an L^2 boundedness criterion) for non-homogeneous singular integrals: Nazarov–Treil–Volberg '03.

Operator estimates via good/bad decomposition

- Suppose we can prove $|\langle Tf_{\text{good}}, g_{\text{good}}\rangle| \leqslant C_{\text{good}}\|f_{\text{good}}\|_2\|g_{\text{good}}\|_2$.
- (Actually very highly nontrivial, but doable!)
- Bad part: "It could be difficult unless it was easy." (A. McIntosh)

$$\begin{aligned} |\langle Tf, g\rangle| &= \mathbb{E}|\langle Tf_{\text{good}}, g_{\text{good}}\rangle + \langle Tf_{\text{bad}}, g_{\text{good}}\rangle + \langle Tf, g_{\text{bad}}\rangle| \\ &\leqslant C_{\text{good}}\|f\|_2\|g\|_2 + \|T\|\mathbb{E}\|f_{\text{bad}}\|_2\|g\|_2 + \|T\|\|f\|_2\mathbb{E}\|g_{\text{bad}}\|_2 \\ &\leqslant \big(C_{\text{good}} + 2\sqrt{\eta}\|T\|\big)\|f\|_2\|g\|_2 \end{aligned}$$

- $\Rightarrow \|T\| \leqslant C_{\text{good}} + 2\sqrt{\eta}\|T\| \Rightarrow \|T\| \leqslant C_{\text{good}}(1-2\sqrt{\eta})^{-1}$.
- $\Rightarrow$ "$T1$ theorem" (= an L^2 boundedness criterion) for non-homogeneous singular integrals: Nazarov–Treil–Volberg '03.
- ($\Rightarrow$ Painlevé's problem on analytic capacity solved by Tolsa '03.)

Variant I: Banach-valued nonhom $T1$ thm (H.'14)

Variant I: Banach-valued nonhom $T1$ thm (H.'14)

- Instead of $\|f\|^2_{L^2(\mu)} = \sum_{Q\in\mathscr{D}} \|\Delta_{Q\dot{+}\beta} f\|_{L^2(\mu)}$,

Variant I: Banach-valued nonhom $T1$ thm (H.'14)

- Instead of $\|f\|^2_{L^2(\mu)} = \sum_{Q\in\mathscr{D}} \|\Delta_{Q\dot{+}\beta} f\|_{L^2(\mu)}$, we have
  $\|f\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)}$, ε_Q random ± 1

Variant I: Banach-valued nonhom $T1$ thm (H.'14)

- Instead of $\|f\|^2_{L^2(\mu)} = \sum_{Q\in\mathscr{D}} \|\Delta_{Q\dot{+}\beta} f\|_{L^2(\mu)}$, we have

  $\|f\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)}$, ε_Q random ± 1

$$\mathbb{E}_\beta \|f_{\text{bad}}\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \mathbb{E}_\beta \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q 1_{\text{bad}}(Q\dot{+}\beta) \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)} \qquad (*)$$

Variant I: Banach-valued nonhom $T1$ thm (H.'14)

- Instead of $\|f\|_{L^2(\mu)}^2 = \sum_{Q\in\mathscr{D}} \|\Delta_{Q\dot{+}\beta} f\|_{L^2(\mu)}$, we have
$\|f\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)}$, ε_Q random ± 1

$$\mathbb{E}_\beta \|f_{\mathrm{bad}}\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \mathbb{E}_\beta \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q 1_{\mathrm{bad}}(Q\dot{+}\beta) \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)} \qquad (*)$$

- H.–Veraar 2009: if Z has **cotype** $q < r$,

Variant I: Banach-valued nonhom $T1$ thm (H.'14)

- Instead of $\|f\|_{L^2(\mu)}^2 = \sum_{Q\in\mathscr{D}} \|\Delta_{Q\dot{+}\beta} f\|_{L^2(\mu)}$, we have
$\|f\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)}$, ε_Q random ± 1

$$\mathbb{E}_\beta \|f_{\mathrm{bad}}\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \mathbb{E}_\beta \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q 1_{\mathrm{bad}}(Q\dot{+}\beta) \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)} \qquad (*)$$

- H.–Veraar 2009: if Z has **cotype** $q < r$, then

$$\mathbb{E}_\varepsilon \Big\| \sum_i \varepsilon_i f_i z_i \Big\|_{L^r(\nu;Z)} \lesssim \sup_i \|f_i\|_{L^r(\nu)} \mathbb{E}_\varepsilon \Big\| \sum_i \varepsilon_i z_i \Big\|_Z$$

Variant I: Banach-valued nonhom $T1$ thm (H.'14)

- Instead of $\|f\|_{L^2(\mu)}^2 = \sum_{Q\in\mathscr{D}} \|\Delta_{Q\dot{+}\beta} f\|_{L^2(\mu)}$, we have
  $$\|f\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)}, \quad \varepsilon_Q \text{ random } \pm 1$$
  $$\mathbb{E}_\beta \|f_{\text{bad}}\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \mathbb{E}_\beta \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q 1_{\text{bad}}(Q\dot{+}\beta) \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)} \qquad (*)$$
- H.–Veraar 2009: if Z has **cotype** $q < r$, then
  $$\mathbb{E}_\varepsilon \Big\| \sum_i \varepsilon_i f_i z_i \Big\|_{L^r(\nu;Z)} \lesssim \sup_i \|f_i\|_{L^r(\nu)} \mathbb{E}_\varepsilon \Big\| \sum_i \varepsilon_i z_i \Big\|_Z$$
- Assuming any **finite cotype** $q < \infty$ for X,

Variant I: Banach-valued nonhom $T1$ thm (H.'14)

- Instead of $\|f\|_{L^2(\mu)}^2 = \sum_{Q\in\mathscr{D}} \|\Delta_{Q\dot{+}\beta} f\|_{L^2(\mu)}$, we have
  $\|f\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)}$, ε_Q random ± 1
  $$\mathbb{E}_\beta \|f_{\text{bad}}\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \mathbb{E}_\beta \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q 1_{\text{bad}}(Q\dot{+}\beta) \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)} \qquad (*)$$
- H.–Veraar 2009: if Z has **cotype** $q < r$, then
  $$\mathbb{E}_\varepsilon \Big\| \sum_i \varepsilon_i f_i z_i \Big\|_{L^r(\nu;Z)} \lesssim \sup_i \|f_i\|_{L^r(\nu)} \mathbb{E}_\varepsilon \Big\| \sum_i \varepsilon_i z_i \Big\|_Z$$
- Assuming any **finite cotype** $q < \infty$ for X, take $r \in (q, \infty)$,

Variant I: Banach-valued nonhom $T1$ thm (H.'14)

- Instead of $\|f\|_{L^2(\mu)}^2 = \sum_{Q\in\mathscr{D}} \|\Delta_{Q\dot{+}\beta} f\|_{L^2(\mu)}$, we have

  $$\|f\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)}, \quad \varepsilon_Q \text{ random } \pm 1$$

  $$\mathbb{E}_\beta \|f_{\mathrm{bad}}\|_{L^p(\mu;X)} \eqsim \mathbb{E}_\varepsilon \mathbb{E}_\beta \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q 1_{\mathrm{bad}}(Q\dot{+}\beta) \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)} \qquad (*)$$

- H.–Veraar 2009: if Z has **cotype** $q < r$, then

  $$\mathbb{E}_\varepsilon \Big\| \sum_i \varepsilon_i f_i z_i \Big\|_{L^r(\nu;Z)} \lesssim \sup_i \|f_i\|_{L^r(\nu)} \mathbb{E}_\varepsilon \Big\| \sum_i \varepsilon_i z_i \Big\|_Z$$

- Assuming any **finite cotype** $q < \infty$ for X, take $r \in (q, \infty)$,

  $$(*) \lesssim \sup_Q (\mathbb{E}_\beta 1_{\mathrm{bad}}(Q\dot{+}\beta)^r)^{1/r} \cdot \mathbb{E}_\varepsilon \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)}$$

  $$\lesssim \eta^{1/r} \cdot \mathbb{E}_\varepsilon \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \Delta_{Q\dot{+}\beta} f \Big\|_{L^p(\mu;X)} \lesssim \eta^{1/r} \|f\|_{L^p(\mu;X)}.$$

Variant II: The A_2 theorem (H.'12)

Variant II: The A_2 theorem (H.'12)

- $\|Tf\|_{L^2(w)} \leqslant [w]_{A_2}\|f\|_{L^2(w)}$, T CZO, $[w]_{A_2} = \sup_Q \langle w\rangle_Q \langle w^{-1}\rangle_Q$.

Variant II: The A_2 theorem (H.'12)

- $\|Tf\|_{L^2(w)} \leqslant [w]_{A_2}\|f\|_{L^2(w)}$, T CZO, $[w]_{A_2} = \sup_Q \langle w \rangle_Q \langle w^{-1} \rangle_Q$.
- NTV-bad cubes again trouble, so write $f = f_{\text{good}} + f_{\text{bad}}$ etc.

Variant II: The A_2 theorem (H.'12)

- $\|Tf\|_{L^2(w)} \leqslant [w]_{A_2}\|f\|_{L^2(w)}$, T CZO, $[w]_{A_2} = \sup_Q \langle w\rangle_Q \langle w^{-1}\rangle_Q$.
- NTV-bad cubes again trouble, so write $f = f_{\text{good}} + f_{\text{bad}}$ etc.
- Desired bound $|\langle Tf_{\text{good}}, g_{\text{good}}\rangle| \leqslant C(r)[w]_{A_2}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}$.

Variant II: The A_2 theorem (H.'12)

- $\|Tf\|_{L^2(w)} \leqslant [w]_{A_2}\|f\|_{L^2(w)}$, T CZO, $[w]_{A_2} = \sup_Q \langle w\rangle_Q \langle w^{-1}\rangle_Q$.
- NTV-bad cubes again trouble, so write $f = f_{\text{good}} + f_{\text{bad}}$ etc.
- Desired bound $|\langle Tf_{\text{good}}, g_{\text{good}}\rangle| \leqslant C(r)[w]_{A_2}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}$.
- r is the parameter in the definition of good and bad.

Variant II: The A_2 theorem (H.'12)

- $\|Tf\|_{L^2(w)} \leqslant [w]_{A_2} \|f\|_{L^2(w)}$, T CZO, $[w]_{A_2} = \sup_Q \langle w \rangle_Q \langle w^{-1} \rangle_Q$.
- NTV-bad cubes again trouble, so write $f = f_{\text{good}} + f_{\text{bad}}$ etc.
- Desired bound $|\langle Tf_{\text{good}}, g_{\text{good}} \rangle| \leqslant C(r)[w]_{A_2} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}$.
- r is the parameter in the definition of good and bad.
- For $\|f_{\text{bad}}\|_{L^2(w)} \leqslant \eta \|f\|_{L^2(w)}$, need $r = r(w)$, thus $C(r) = C(w)$!

Variant II: The A_2 theorem (H.'12)

- $\|Tf\|_{L^2(w)} \leqslant [w]_{A_2}\|f\|_{L^2(w)}$, T CZO, $[w]_{A_2} = \sup_Q \langle w\rangle_Q \langle w^{-1}\rangle_Q$.
- NTV-bad cubes again trouble, so write $f = f_{\text{good}} + f_{\text{bad}}$ etc.
- Desired bound $|\langle Tf_{\text{good}}, g_{\text{good}}\rangle| \leqslant C(r)[w]_{A_2}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}$.
- r is the parameter in the definition of good and bad.
- For $\|f_{\text{bad}}\|_{L^2(w)} \leqslant \eta\|f\|_{L^2(w)}$, need $r = r(w)$, thus $C(r) = C(w)$!
- New idea: Average bad cubes out altogether.

Variant II: The A_2 theorem (H.'12)

- $\|Tf\|_{L^2(w)} \leqslant [w]_{A_2}\|f\|_{L^2(w)}$, T CZO, $[w]_{A_2} = \sup_Q \langle w \rangle_Q \langle w^{-1} \rangle_Q$.
- NTV-bad cubes again trouble, so write $f = f_{\text{good}} + f_{\text{bad}}$ etc.
- Desired bound $|\langle Tf_{\text{good}}, g_{\text{good}} \rangle| \leqslant C(r)[w]_{A_2}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}$.
- r is the parameter in the definition of good and bad.
- For $\|f_{\text{bad}}\|_{L^2(w)} \leqslant \eta\|f\|_{L^2(w)}$, need $r = r(w)$, thus $C(r) = C(w)$!
- New idea: Average bad cubes out altogether.

$$\begin{aligned}
\mathbb{E}f_{\text{good}} &= \mathbb{E}\sum_{Q\in\mathscr{D}} 1_{\text{good}}(Q\dot{+}\beta)\Delta_{Q\dot{+}\beta}f \\
&= \sum_{Q\in\mathscr{D}} \mathbb{E}(1_{\text{good}}(Q\dot{+}\beta))\mathbb{E}(\Delta_{Q\dot{+}\beta}f) \quad \text{indep!} \\
&= \pi_{\text{good}}\sum_{Q\in\mathscr{D}} \mathbb{E}(\Delta_{Q\dot{+}\beta}f) \quad \pi_{\text{good}} := \mathbb{E}(\Delta_{Q\dot{+}\beta}f) \text{ const} \\
&= \pi_{\text{good}}\mathbb{E}f = \pi_{\text{good}}f.
\end{aligned}$$

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

- $T(\cdot\mu) : L^2(\mu) \to L^2(\nu)$ bounded $\Rightarrow \|T(1_Q\mu)\|_{L^2(\nu)} \lesssim \mu(Q)^{1/2}$

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

- $T(\cdot\mu) : L^2(\mu) \to L^2(\nu)$ bounded $\Rightarrow \|T(1_Q\mu)\|_{L^2(\nu)} \lesssim \mu(Q)^{1/2}$
- Sometimes a nontrivial "$\Leftarrow$" holds: "testing condition".

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

- $T(\cdot\mu) : L^2(\mu) \to L^2(\nu)$ bounded $\Rightarrow \|T(1_Q\mu)\|_{L^2(\nu)} \lesssim \mu(Q)^{1/2}$
- Sometimes a nontrivial "$\Leftarrow$" holds: "testing condition".
- Sawyer'82: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \forall Q$:

$$\|M(1_Q u)\|_{L^2(v)} \lesssim u(Q)^{1/2} \qquad (*)$$

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

- $T(\cdot\mu) : L^2(\mu) \to L^2(\nu)$ bounded $\Rightarrow \|T(1_Q\mu)\|_{L^2(\nu)} \lesssim \mu(Q)^{1/2}$
- Sometimes a nontrivial "$\Leftarrow$" holds: "testing condition".
- Sawyer'82: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \forall Q$:

$$\|M(1_Q u)\|_{L^2(v)} \lesssim u(Q)^{1/2} \qquad (*)$$

- But if u is quite general, even this testing may be hard to verify.

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

- $T(\cdot\mu) : L^2(\mu) \to L^2(\nu)$ bounded $\Rightarrow \|T(1_Q\mu)\|_{L^2(\nu)} \lesssim \mu(Q)^{1/2}$
- Sometimes a nontrivial "$\Leftarrow$" holds: "testing condition".
- Sawyer'82: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \forall Q$:

$$\|M(1_Q u)\|_{L^2(v)} \lesssim u(Q)^{1/2} \qquad (*)$$

- But if u is quite general, even this testing may be hard to verify.
- Would like to test with even weaker $\|M(1_Q u)\|_{L^p(v)} \lesssim u(\rho Q)^{1/p}$,

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

- $T(\cdot\mu) : L^2(\mu) \to L^2(\nu)$ bounded $\Rightarrow \|T(1_Q\mu)\|_{L^2(\nu)} \lesssim \mu(Q)^{1/2}$
- Sometimes a nontrivial "$\Leftarrow$" holds: "testing condition".
- Sawyer'82: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \forall Q$:

$$\|M(1_Q u)\|_{L^2(v)} \lesssim u(Q)^{1/2} \qquad (*)$$

- But if u is quite general, even this testing may be hard to verify.
- Would like to test with even weaker $\|M(1_Q u)\|_{L^p(v)} \lesssim u(\rho Q)^{1/p}$, or still weaker: only with those Q such that $u(\rho Q) \lesssim u(Q)$!

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

- $T(\cdot\mu) : L^2(\mu) \to L^2(\nu)$ bounded $\Rightarrow \|T(1_Q\mu)\|_{L^2(\nu)} \lesssim \mu(Q)^{1/2}$
- Sometimes a nontrivial "$\Leftarrow$" holds: "testing condition".
- Sawyer'82: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \forall Q$:

$$\|M(1_Q u)\|_{L^2(v)} \lesssim u(Q)^{1/2} \qquad (*)$$

- But if u is quite general, even this testing may be hard to verify.
- Would like to test with even weaker $\|M(1_Q u)\|_{L^p(v)} \lesssim u(\rho Q)^{1/p}$, or still weaker: only with those Q such that $u(\rho Q) \lesssim u(Q)$!
- Late 2018: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow$

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

- $T(\cdot\mu) : L^2(\mu) \to L^2(\nu)$ bounded $\Rightarrow \|T(1_Q\mu)\|_{L^2(\nu)} \lesssim \mu(Q)^{1/2}$
- Sometimes a nontrivial "$\Leftarrow$" holds: "testing condition".
- Sawyer'82: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \forall Q$:

$$\|M(1_Q u)\|_{L^2(v)} \lesssim u(Q)^{1/2} \qquad (*)$$

- But if u is quite general, even this testing may be hard to verify.
- Would like to test with even weaker $\|M(1_Q u)\|_{L^p(v)} \lesssim u(\rho Q)^{1/p}$, or still weaker: only with those Q such that $u(\rho Q) \lesssim u(Q)$!
- Late 2018: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \langle u\rangle_Q \langle v\rangle_Q \leqslant C \ \forall Q$ &

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

- $T(\cdot\mu) : L^2(\mu) \to L^2(\nu)$ bounded $\Rightarrow \|T(1_Q\mu)\|_{L^2(\nu)} \lesssim \mu(Q)^{1/2}$
- Sometimes a nontrivial "$\Leftarrow$" holds: "testing condition".
- Sawyer'82: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \forall Q$:

$$\|M(1_Q u)\|_{L^2(v)} \lesssim u(Q)^{1/2} \qquad (*)$$

- But if u is quite general, even this testing may be hard to verify.
- Would like to test with even weaker $\|M(1_Q u)\|_{L^p(v)} \lesssim u(\rho Q)^{1/p}$, or still weaker: only with those Q such that $u(\rho Q) \lesssim u(Q)$!
- Late 2018: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \langle u\rangle_Q \langle v\rangle_Q \leqslant C \ \forall Q$ & $(*)$
  - if $u(\rho Q) \lesssim u(Q)$. (Li-Sawyer)

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

- $T(\cdot\mu) : L^2(\mu) \to L^2(\nu)$ bounded $\Rightarrow \|T(1_Q\mu)\|_{L^2(\nu)} \lesssim \mu(Q)^{1/2}$
- Sometimes a nontrivial "$\Leftarrow$" holds: "testing condition".
- Sawyer'82: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \forall Q$:

$$\|M(1_Q u)\|_{L^2(v)} \lesssim u(Q)^{1/2} \qquad (*)$$

- But if u is quite general, even this testing may be hard to verify.
- Would like to test with even weaker $\|M(1_Q u)\|_{L^p(v)} \lesssim u(\rho Q)^{1/p}$, or still weaker: only with those Q such that $u(\rho Q) \lesssim u(Q)$!
- Late 2018: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \langle u\rangle_Q \langle v\rangle_Q \leqslant C\ \forall Q\ \&\ (*)$
  - if $u(\rho Q) \lesssim u(Q)$. (Li-Sawyer)
  - if $u(P) \lesssim u(Q)$, *some* $P \subset Q$ w/ $\ell(P) = \rho\ell(Q)$ (Chen–Lacey)

Variant III (w/ Kangwei Li & E. Sawyer, in progr.)

- $T(\cdot\mu) : L^2(\mu) \to L^2(\nu)$ bounded $\Rightarrow \|T(1_Q\mu)\|_{L^2(\nu)} \lesssim \mu(Q)^{1/2}$
- Sometimes a nontrivial "$\Leftarrow$" holds: "testing condition".
- Sawyer'82: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \forall Q$:

$$\|M(1_Q u)\|_{L^2(v)} \lesssim u(Q)^{1/2} \qquad (*)$$

- But if u is quite general, even this testing may be hard to verify.
- Would like to test with even weaker $\|M(1_Q u)\|_{L^p(v)} \lesssim u(\rho Q)^{1/p}$, or still weaker: only with those Q such that $u(\rho Q) \lesssim u(Q)$!
- Late 2018: $\|M(fu)\|_{L^2(v)} \lesssim \|f\|_{L^2(u)} \Leftrightarrow \langle u\rangle_Q \langle v\rangle_Q \leqslant C\ \forall Q\ \&\ (*)$
  - if $u(\rho Q) \lesssim u(Q)$. (Li-Sawyer)
  - if $u(P) \lesssim u(Q)$, *some* $P \subset Q$ w/ $\ell(P) = \rho\ell(Q)$ (Chen–Lacey)
- in progr: better proof of Li-Sawyer.

Variant III proof detail: random Whitney cubes

Variant III proof detail: random Whitney cubes

- Whitney decomposition of open $\Omega \subset \mathbb{R}^d$: a cover $\mathscr{W} \subset \mathscr{D}$ s.t.

Variant III proof detail: random Whitney cubes

- Whitney decomposition of open $\Omega \subset \mathbb{R}^d$: a cover $\mathscr{W} \subset \mathscr{D}$ s.t.

$$\operatorname{dist}(Q, \Omega^c) \simeq \ell(Q) \quad \forall Q \in \mathscr{W} \qquad \text{w/ controlled overlap.}$$

Variant III proof detail: random Whitney cubes

- Whitney decomposition of open $\Omega \subset \mathbb{R}^d$: a cover $\mathscr{W} \subset \mathscr{D}$ s.t.

$$\mathrm{dist}(Q, \Omega^c) \simeq \ell(Q) \quad \forall Q \in \mathscr{W} \qquad \text{w/ controlled overlap.}$$

- Usual way: take maximal $Q \in \mathscr{D}$ s.t. $\mathrm{dist}(Q, \Omega^c) > C\ell(Q)$.

Variant III proof detail: random Whitney cubes

- Whitney decomposition of open $\Omega \subset \mathbb{R}^d$: a cover $\mathscr{W} \subset \mathscr{D}$ s.t.

$$\operatorname{dist}(Q, \Omega^c) \simeq \ell(Q) \quad \forall Q \in \mathscr{W} \qquad \text{w/ controlled overlap.}$$

- Usual way: take maximal $Q \in \mathscr{D}$ s.t. $\operatorname{dist}(Q, \Omega^c) > C\ell(Q)$.
- Then $\operatorname{dist}(\hat{Q}, \Omega^c) \leqslant C\ell(Q)$, thus $\operatorname{dist}(Q, \Omega^c) \leqslant (C + \sqrt{d})\ell(Q)$.

Variant III proof detail: random Whitney cubes

- Whitney decomposition of open $\Omega \subset \mathbb{R}^d$: a cover $\mathscr{W} \subset \mathscr{D}$ s.t.

$$\operatorname{dist}(Q, \Omega^c) \simeq \ell(Q) \quad \forall Q \in \mathscr{W} \qquad \text{w/ controlled overlap.}$$

- Usual way: take maximal $Q \in \mathscr{D}$ s.t. $\operatorname{dist}(Q, \Omega^c) > C\ell(Q)$.
- Then $\operatorname{dist}(\hat{Q}, \Omega^c) \leqslant C\ell(Q)$, thus $\operatorname{dist}(Q, \Omega^c) \leqslant (C + \sqrt{d})\ell(Q)$.
- Problem with cubes $P \in \mathscr{D}^\beta$: to see if P is "maximal", need to know $\hat{P}$, which also depends on β_j for $2^j = \ell(P)$.

Variant III proof detail: random Whitney cubes

- Whitney decomposition of open $\Omega \subset \mathbb{R}^d$: a cover $\mathscr{W} \subset \mathscr{D}$ s.t.

$$\operatorname{dist}(Q, \Omega^c) \simeq \ell(Q) \quad \forall Q \in \mathscr{W} \qquad \text{w/ controlled overlap.}$$

- Usual way: take maximal $Q \in \mathscr{D}$ s.t. $\operatorname{dist}(Q, \Omega^c) > C\ell(Q)$.
- Then $\operatorname{dist}(\hat{Q}, \Omega^c) \leqslant C\ell(Q)$, thus $\operatorname{dist}(Q, \Omega^c) \leqslant (C + \sqrt{d})\ell(Q)$.
- Problem with cubes $P \in \mathscr{D}^\beta$: to see if P is "maximal", need to know $\hat{P}$, which also depends on β_j for $2^j = \ell(P)$.
- Events "P is a Whitney cube" and "P is good" not independent!

Variant III proof detail: random Whitney cubes

- Whitney decomposition of open $\Omega \subset \mathbb{R}^d$: a cover $\mathscr{W} \subset \mathscr{D}$ s.t.

$$\operatorname{dist}(Q, \Omega^c) \simeq \ell(Q) \quad \forall Q \in \mathscr{W} \qquad \text{w/ controlled overlap.}$$

- Usual way: take maximal $Q \in \mathscr{D}$ s.t. $\operatorname{dist}(Q, \Omega^c) > C\ell(Q)$.
- Then $\operatorname{dist}(\hat{Q}, \Omega^c) \leqslant C\ell(Q)$, thus $\operatorname{dist}(Q, \Omega^c) \leqslant (C + \sqrt{d})\ell(Q)$.
- Problem with cubes $P \in \mathscr{D}^\beta$: to see if P is "maximal", need to know $\hat{P}$, which also depends on β_j for $2^j = \ell(P)$.
- Events "P is a Whitney cube" and "P is good" not independent! $\Rightarrow$ very complicated computations.

Variant III proof detail: random Whitney cubes

- Whitney decomposition of open $\Omega \subset \mathbb{R}^d$: a cover $\mathscr{W} \subset \mathscr{D}$ s.t.

$$\operatorname{dist}(Q, \Omega^c) \simeq \ell(Q) \quad \forall Q \in \mathscr{W} \qquad \text{w/ controlled overlap.}$$

- Usual way: take maximal $Q \in \mathscr{D}$ s.t. $\operatorname{dist}(Q, \Omega^c) > C\ell(Q)$.
- Then $\operatorname{dist}(\hat{Q}, \Omega^c) \leqslant C\ell(Q)$, thus $\operatorname{dist}(Q, \Omega^c) \leqslant (C + \sqrt{d})\ell(Q)$.
- Problem with cubes $P \in \mathscr{D}^\beta$: to see if P is "maximal", need to know $\hat{P}$, which also depends on β_j for $2^j = \ell(P)$.
- Events "P is a Whitney cube" and "P is good" not independent! $\Rightarrow$ very complicated computations.
- Sol'n I: Redefine $\mathscr{W}$ as all Q s.t. $a\ell(Q) < \operatorname{dist}(Q, \Omega^c) < b\ell(Q)$ (partition repl. by bounded overlap; independence restored)

Variant III proof detail: random Whitney cubes

- Whitney decomposition of open $\Omega \subset \mathbb{R}^d$: a cover $\mathscr{W} \subset \mathscr{D}$ s.t.

$$\operatorname{dist}(Q, \Omega^c) \simeq \ell(Q) \quad \forall Q \in \mathscr{W} \qquad \text{w/ controlled overlap.}$$

- Usual way: take maximal $Q \in \mathscr{D}$ s.t. $\operatorname{dist}(Q, \Omega^c) > C\ell(Q)$.
- Then $\operatorname{dist}(\hat{Q}, \Omega^c) \leqslant C\ell(Q)$, thus $\operatorname{dist}(Q, \Omega^c) \leqslant (C + \sqrt{d})\ell(Q)$.
- Problem with cubes $P \in \mathscr{D}^\beta$: to see if P is "maximal", need to know $\hat{P}$, which also depends on β_j for $2^j = \ell(P)$.
- Events "P is a Whitney cube" and "P is good" not independent! $\Rightarrow$ very complicated computations.
- Sol'n I: Redefine $\mathscr{W}$ as all Q s.t. $a\ell(Q) < \operatorname{dist}(Q, \Omega^c) < b\ell(Q)$ (partition repl. by bounded overlap; independence restored)
- Sol'n II: Redefine "Q good" as "$\hat{Q}$ good"! (probably works)

The end

Thanks for your attention!