# Contribution to an Open Problem of Harkness and Shantaram, the Limit of Families Obtained by the Stationary Excess Operator

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# Some setting

- All random variables are nonnegative.
- $\mathbb{T} = \mathbb{N}$ or $[0, \infty)$. A subsequence of $(X_t)_{t\in\mathbb{T}}$, is a sequence $(X_{t(n)})_{n\in\mathbb{N}}$ with function $t : \mathbb{N} \to \mathbb{T}$ s.t. $t(n) \nearrow \infty$.
- For $a, b > 0$,, $\mathfrak{b}_{a,b}$ and $\mathfrak{g}_a$ denote r.v.'s with respectively beta and Gamma distribution . It is well known that

$$\mathfrak{g}_a \stackrel{d}{=} \mathfrak{b}_{a,b}\, \mathfrak{g}_{a+b} \quad \text{and} \quad \mathfrak{b}_{a,b+c} \stackrel{d}{=} \mathfrak{b}_{a,b}\, \mathfrak{b}_{a+b,c}, \qquad a, b, c > 0\,.$$

- The Mellin transform:

$$\begin{aligned} \mathcal{M}_X(\lambda) &= \mathbb{E}[X^\lambda], \quad \text{for } \lambda \in \mathcal{D}_X \subseteq \mathbb{C} \\ \lambda_X &= \sup\{\lambda \in \mathbb{R},\ \mathbb{E}[X^\lambda] < \infty\}. \end{aligned}$$

# 1. Harkness and Shantaram's problem

# The problem

The stationary excess operator is given by

$$\mathbb{P}(\mathcal{E}_1(X) \leq x) = \frac{1}{\mathbb{E}[X]} \int_0^x \mathbb{P}(X > u)\, du, \quad x \geq 0$$

$$\mathcal{E}_{n+1} = \mathcal{E}_1 \circ \mathcal{E}_n, \quad n \in \mathbb{N}.$$

**Theorem (Harkness and Shantaram, 1969)**

*If there exists a sequence* $c_n$ *s.t.* $\limsup_{n\to\infty} c_{n+1}/c_n \in [1, \infty)$ *and*

$$Z_n := \frac{\mathcal{E}_n(X)}{c_n} \xrightarrow{d} Z_\infty, \quad \text{as } n \to \infty,$$

*then, necessarily* $\lim_{n\to\infty} c_{n+1}/c_n = L \in [1, \infty)$ *and*

$$\lim_{n\to\infty} \mathbb{E}[Z_n^k] = \mathbb{E}[Z_\infty^k] \in (0, \infty), \quad \forall\, k \in \mathbb{N}.$$

# Natural questions

1) What is the set of possible distributions for $Z_\infty$?
Arratia, Goldstein and Kochman (2015), van Beek and Braat (1973), Garcia (2009), Shantaram and Harkness (1972), Pakes (1996), Vardi, Shepp and Logan (1981). Their approach was mainly based on the identity in law:

$$\boxed{Z_\infty \stackrel{d}{=} U_L\,(Z_\infty)_{(1)},} \qquad (Z_\infty)_{(1)} \stackrel{d}{=} x\,\mathbb{P}(Z \in dx)/\,\mathbb{E}[Z].$$

2) What additional information on the distribution of $Z_\infty$ can we obtain if we study the continuous scheme

$$Z_t = \frac{\mathcal{E}_t(X)}{c_t}, \quad t \in [0,\infty)?$$

$\mathcal{E}_t$ = continuous time stationary-excess operator given by

$$\boxed{\mathbb{P}(\mathcal{E}_t(X) > x) = \frac{t}{\mathbb{E}[X^t]}\int_x^\infty (u-x)^{t-1}\,\mathbb{P}(X>u)\,du, \quad x \geq 0\,.}$$

# More on contituous time stationary-excess operator

1) $(\mathcal{E}_t)_{t\in\mathbb{T}}$ forms a semigroup ($\mathcal{E}_t \circ \mathcal{E}_s = \mathcal{E}_{t+s}$) and

$$\boxed{\mathcal{E}_t(X) :\overset{d}{=} \mathfrak{b}_t\, X_{(t)},}$$

where $\mathfrak{b}_t$ is independent from $X_{(t)}$, $\mathfrak{b}_t \overset{d}{=} \mathfrak{b}_{1,t}$ and

$$X_{(t)} \overset{d}{=} x^t\ \mathbb{P}(X \in dx)/\,\mathbb{E}[X^t].$$

2) Nice behavior for Size Biazing:

- $\mathbb{E}[g(X(t))] = \frac{\mathbb{E}[X^t g(X)]}{\mathbb{E}[X^t]}$
- $\left(X_{(s)}\right)_{(t)} \overset{d}{=} X_{(s+t)} \overset{d}{=} \left(X_{(t)}\right)_{(s)}$
- $(XY)_{(t)} \overset{d}{=} X_{(t)}Y_{(t)}$, if $X$, $Y$ and $X_{(t)}$, $Y_{(t)}$ independent.

# More on continuous time stationary excess operator

3) Link with t-monotone functions:

**Definition**

Let $t \in (0,\infty)$. A function $f : (0,\infty) \to [0,\infty)$ is $t$-monotone if

$$f(x) = d + \int_{(0,\infty)} (u-x)_+^{t-1}\, \nu(du), \quad c \geq 0,\, x > 0.$$

**Proposition**

*Let $f$ be a $t$-monotone p.d.f of a r.v. $Z > 0$ $(d = 0)$.*

*i)* $\nu(0,\infty) < \infty$ *IFF* $Z \stackrel{d}{=} \mathfrak{b}_t\, X_{(t)}$.

*ii)* $\looparrowright$ *Alternative proof for Bernstein's characterization of* $\mathcal{CM}$.

# 2. The Normal Limit Theorem

# Reformulation of the problem of Harkness and Shantaram

Recall $Z_t = \frac{\mathcal{E}_t(X)}{c_t}, \quad t \in \mathbb{T} = \mathbb{N} \textit{ or } [0,\infty)$.

- Convergence along the continuous scheme $\{\mathbb{T} = [0,\infty)\}$
$\Longrightarrow$ Convergence along the discrete scheme $\{\mathbb{T} = \mathbb{N}\}$ !

Are the two convergence schemes equivalent?

- What are the NSC for

$$Z_t \overset{d}{=} \frac{1}{c_t}\mathfrak{b}_t\, X_{(t)} \ \xrightarrow{d}\ Z_\infty \quad \text{when } t \in \mathbb{T} \text{ and } t \to \infty?$$

- Reformulation: Since $t\mathfrak{b}_t \xrightarrow{d} \mathfrak{e}$, where $\mathfrak{e}$ is exponentially distributed, it is natural to find NSC on $\rho_t = t\,c_t$, s.t.

$$X_t := \frac{X_{(t)}}{\rho_t} \ \xrightarrow{d}\ X_\infty\,!$$

- Necessarily: $Z_\infty \overset{d}{=} \mathfrak{e}\, X_\infty$, where $\mathfrak{e}$ is independent from $X_\infty$.

## Theorem (The Normal Limit Theorem)

*1) The following statements are equivalent:*

*(i)* $Z_n \xrightarrow{d} Z_\infty$ *and* $\mathbb{E}[Z_\infty^{\lambda_0}] < \infty$ *for some* $\lambda_0 > 0$*;*

*(ii)* $Z_t \xrightarrow{d} Z_\infty$ *and* $\mathbb{E}[Z_\infty^{\lambda_0}] < \infty$ *for some* $\lambda_0 > 0$*;*

*(iii)* $X_n \xrightarrow{d} X_\infty$ *and* $\mathbb{E}[X_\infty^{\lambda_0}] < \infty$ *for some* $\lambda_0 > 0$*;*

*(iv)* $X_t \xrightarrow{d} X_\infty$ *and* $\mathbb{E}[X_\infty^{\lambda_0}] < \infty$ *for some* $\lambda_0 > 0$*;*

*(v)* $\mathbb{E}[X_t^\lambda] \to \mathbb{E}[X_\infty^\lambda]$, *for all* $\lambda \in [0, \infty)$.

*(vi)* $X_t \xrightarrow{d} X_\infty$ *and* $\limsup_{t\to\infty} \frac{\rho_{t+s}}{\rho_t} < \infty$ *for some* $s \in \mathbb{T} \setminus \{0\}$.

*2) Under any of the latter, there exists* $c \geq 0$*, s.t. for all* $s \geq 0$*,*

$$\rho_t \overset{+\infty}{\sim} \mathbb{E}[X_\infty]\,\mathbb{E}[X^{t+1}]/\,\mathbb{E}[X^t] \text{ and } \lim_{t\to\infty} \frac{\rho_{t+s}}{\rho_t} = e^{cs}$$

$$Z_\infty \overset{d}{=} \mathfrak{e}\, X_\infty \quad \text{where } \mathfrak{e} \perp X_\infty$$

$$\overset{d}{=} e^{-cs}\, \mathfrak{b}_s\, (Z_\infty)_{(s)} \quad \text{where } \mathfrak{b}_s \perp (Z_\infty)_{(s)}$$

$$\log X_\infty \overset{d}{=} \textit{Normal}\,(\log \mathbb{E}[X_\infty] - \frac{c}{2}, \frac{c}{2}).$$

# An non-trivial Example

Let $g : (0, \infty) \to \mathbb{R}$, recall given by $\Delta_a g(x) := g(x+a) - g(x)$.
$g$ is said monotone of order $k \in \mathbb{N}$, if

$$(-1)^k \, \Delta_{a_1} \Delta_{a_2} \cdots \Delta_{a_k} g \leq 0, \quad \forall \, a_1, a_2, \cdots a_k > 0, \; k \in \mathbb{N} \setminus \{0\}.$$

- $(-1)^k \, g^{(k)} \geq 0$ implies that $g$ is monotone of order $k$;
- Choose $\alpha = \mathbb{E}[X_\infty]$, and define for $t \geq 0$

$$g_{\scriptscriptstyle X}(t) = \log \mathbb{E}[X^t] \quad \text{and} \quad \rho_t = \alpha \; \mathbb{E}[X_{(t)}] = \alpha \frac{\mathbb{E}[X^{t+1}]}{\mathbb{E}[X^t]} = \alpha \exp \Delta_1 g_{\scriptscriptstyle X}(t);$$

- We already know that that $g_{\scriptscriptstyle X}$ is convex, that $t \mapsto \rho_t \nearrow$ and then

$$\frac{\rho_{t+s}}{\rho_t} = \frac{\mathbb{E}[X_{(t+s)}]}{\mathbb{E}[X_{(t)}]} = \exp \Delta_1 \Delta_s g_{\scriptscriptstyle X}(t) \leq 1;$$

- If $g_{\scriptscriptstyle X}$ is monotone of order 3 ( $g'_{\scriptscriptstyle X}$ is concave), then $t \mapsto \rho_{t+s}/\rho_t \searrow$;
- $\lim_{t\to\infty} \rho_{t+s}/\rho_t = e^{cs}$ for some $c \geq 0$.

## Example

1) If $g'_X$ is a concave function, then $X$ satisfies the Normal Limit Theorem.

2)For instance, $\log X$ is an infinite divisible random variable such that its Lévy exponent $g_X = \log \mathcal{M}_X$ has the form

$$g_X(\lambda) = d\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x \mathbb{1}_{x \le 1})\, \pi(dx), \quad \lambda \ge 0,$$

with $d \in \mathbb{R}$, $\sigma \ge 0$. Then $g'_X$ is concave and for $t$, $s > 0$,

$$\Delta_1 \Delta_s g(t) = \sigma^2 s + \int_{(0,\infty)} e^{-tx}(1 - e^{-x})(1 - e^{-sx})\, \pi(dx),$$

and $\lim_{t\to\infty} \rho_{t+s}/\rho_t = e^{\sigma^2 s}$

# 3. The Tools

# The Mellin transform, refined properties

Recall $\lambda_X = \sup\{\lambda \in \mathbb{R}, \mathcal{M}_X(\lambda) := \mathbb{E}[X^\lambda] < \infty\}$.

**Proposition**

*1) If $\lambda_X > 0$, then, on $[0, \lambda_X)$, the function $\mathcal{M}_X$ is log-convex and strictly log-convex.*

*2) Assume $\lambda_X > 0$. For every $\lambda \in (0, \lambda_X)$, the function $t \mapsto \mathcal{M}_X(\lambda + t)/\mathcal{M}_X(t)$ is nondecreasing on $[0, \lambda_X - \lambda)$.*

*3) Everything is strict if $X$ is non-deterministic.*

**Lemma**

*1)* **Widder's Theorem:** *Assume $M_X = M_Y$ on some complex strip $(\alpha, \beta) \subset \mathbb{C}$, then $X \overset{d}{=} Y$.*

*2)* **Improvement:** *The same holds if $(\alpha, \beta) \subset \mathbb{R}$.*

# Convergence of families of Mellin transforms

**Definition (Billingsley)**

(i) $(X_n)_{n\in\mathbb{N}}$ is tight if $\lim_{x\to\infty}\sup_{n\in\mathbb{N}}\mathbb{P}(|X_n|>x)=0.$

(ii) $(X_n)_{n\in\mathbb{N}}$ is UI if $\lim_{x\to\infty}\sup_{n\in\mathbb{N}}\mathbb{E}[|X_n|\mathbb{1}_{X_n>x}]=0.$

**Definition (Extension)**

(i) $(X_t)_{t\in\mathbb{T}}$ is ultimately tight if $\lim_{x\to\infty}\limsup_{t\in\mathbb{T}}\mathbb{P}(X_t>x)=0.$

(ii) $(X_t)_{t\in\mathbb{T}}$ is $\lambda$-UI if $\lim_{x\to\infty}\limsup_{t\in\mathbb{T}}\mathbb{E}[X_t^{\lambda}\mathbb{1}_{X_t>x}]=0.$

(iii) $X_t\xrightarrow{d}X_\infty$, if $X_{t(n)}\xrightarrow{d}X_\infty$, for all $(t(n))_n\subset\mathbb{T}$.

# Convergence of families of Mellin transforms

**Theorem**

*1) Let $X_\infty \geq 0$ and $\lambda_0 > 0$. The following assertions are equivalent, as $t \to \infty$:*

*(i)* $X_t \xrightarrow{d} X_\infty$ *and* $(X_t)_{t\in\mathbb{T}}$ *is* $\lambda_0$*-UI;*

*(ii)* $X_t \xrightarrow{d} X_\infty$ *and* $\mathbb{E}[X_t^{\lambda_0}] \to \mathbb{E}[X_\infty^{\lambda_0}] < \infty$*;*

*(iii)* $\mathbb{E}[X_t^{\lambda}] \to \mathbb{E}[X_\infty^{\lambda}]] < \infty,\ \forall\lambda \in [0, \lambda_0]$.

*2) Assume that* $\lim_{t\to\infty} \mathbb{E}[X_t^{\lambda}] = f(\lambda),\ \lambda \in [\lambda_1, \lambda_0]$. *Then (iii) holds and* $f$ *is well defined on* $[0, \lambda_0]$ *by* $f(\lambda) = \mathbb{E}[X_\infty^{\lambda}]$.

# Convergence of families of Mellin transforms

**Corollary (Simplification)**

*Let* $(U_t)_{t\in\mathbb{T}}$ , $(V_t)_{t\in\mathbb{T}}$ *and* $(W_t)_{t\in\mathbb{T}}$ *s.t.* $U_t$ *and* $V_t$ *are independent and there exists* $\lambda_0 > 0$ *such that* $(W_t)_{t\in\mathbb{T}}$ *is* $\lambda_0 - UI$. *Then,*

$$W_t \overset{d}{=} U_t\, V_t, \quad W_t \xrightarrow{d} W_\infty \quad \textit{and} \quad V_t \xrightarrow{d} V_\infty$$

*yields*

$$U_t \xrightarrow{d} U_\infty \quad \textit{and} \quad \mathbb{E}[U_\infty^\lambda] = \frac{\mathbb{E}[W_\infty^\lambda]}{\mathbb{E}[V_\infty^\lambda]}, \quad \lambda \in [0, \lambda_0].$$

Application: $t\mathfrak{b}_t \xrightarrow{d} \mathfrak{e}$, where $\mathfrak{e} \overset{d}{=} Exp(1)$, choose

$$(U_t, V_t, W_t) = (t\,\mathfrak{b}_t, \frac{X_{(t)}}{\rho_t}, Z_t), \quad \text{with } \rho_t = t\,c_t.$$

and assume $\mathbb{E}[Z_\infty^{\lambda_0}] < \infty$ for some $\lambda_0 \in \mathbb{T} \setminus \{0\}$.

# Bardzo dziękuję za uwagę!

# Thank you for your attention !

# Some References


1. Garcia, M., L.: *Characterization of distributions with the length-bias scaling property*. Elect. Comm. in Probab. **14**, 186–191, 2009.
2. Harkness, W. L., Shantaram, R.: *Convergence of a sequence of transformations of distribution functions*. Pacific Journal of Mathematics, **31**(2), 1969.
3. Pakes, A.G., Sapatinas, T., Fosam, E. B.: *Characterizations, length-biasing, and infinite divisibility*. Statistical Paper **37**, 53–69, 1996.
4. Shantaram, R., Harkness, W. L.: *On a Certain Class of Limit Distributions*. Ann. Math. Statist. **43**(6), 2067–2071, 1972.
5. Schilling, R.L.; Song, R.; Vondraček, Z.: Bernstein Functions. Theory and applications. Second edition, de Gruyter Studies in Mathematics, 37. Walter de Gruyter & Co., Berlin, 2012.
6. Widder, D.V.: The Laplace Transform. Princeton University Press, 1946.