A framework for a nonlinear nonlocal diffusion equation

Grzegorz Karch **Moritz Kassmann** **Miłosz Krupski**

Fractional Laplacian

$$(-\Delta)^s v(x) = C(\alpha) \int\limits_{\mathbb{R}^d} \big(v(x) - v(y)\big) \frac{1}{|x-y|^{d+2s}}\, dy$$

with

$$s \in (0,1) \qquad \alpha = 2s$$

The integral is understood the sense of the Cauchy principal value, if

$$s \in \left[\frac{1}{2}, 1\right).$$

Lévy operator

$$\mathcal{L}v(x) = \int_{\mathbb{R}^d} \big(v(x) - v(y)\big)\, d\mu(x-y)$$

μ is a nonnegative symmetric Radon measure on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\int_{|y|\leqslant 1} |y|^2 \, d\mu(y) + \int_{|y|>1} d\mu(y) < \infty.$$

Fractional Laplacian:

$$\mu(x-y) = \frac{1}{|x-y|^{n+2s}}$$

The integral is understood the sense of the Cauchy principal value, if μ is too singular.

The Cauchy problem

$$u_t + \mathcal{L}u = 0, \qquad u(0) = u_0.$$

Theorem

For every initial datum $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$, the Cauchy problem has a unique weak solution

$$u \in L^\infty\big([0,\infty), L^1 \cap L^\infty(\mathbb{R}^d)\big).$$

This solution has the following properties

- *mass is conserved:* $\int_{\mathbb{R}^d} u(t,x)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx$ *for all* $t \geqslant 0$,
- L^p*-norm is non-increasing:* $\|u(t)\|_p \leqslant \|u_0\|_p$ *for all* $p \in [1,\infty]$ *and* $t \geqslant 0$,
- *if* u_0 *is nonnegative then the corresponding weak solution is nonnegative for almost all* $x \in \mathbb{R}^d$ *and* $t \geqslant 0$.

Nonlinear Cauchy problem

Fractional porous medium equation

$$u_t + (-\Delta)^s f(u) = 0, \qquad u(0) = u_0.$$

See papers by
{de Pablo, Quirós, Rodríguez, Vázquez}
{del Teso, Endal, Jakobsen}.

$$\Big((-\Delta)^s f(u)\Big)(x) = \int_{\mathbb{R}^d} \frac{f(u(x)) - f(u(y))}{|x-y|^{d+2s}}\, dy$$

$$= \int_{\mathbb{R}^d} (u(x) - u(y)) \left(\frac{f(u(x)) - f(u(y))}{u(x) - u(y)} \mu(|x-y|) \right) dy$$

Fractional p-Laplacian

$$\partial_t u + \int_{\mathbb{R}^d} \frac{\Phi\big(u(x) - u(y)\big)}{\big|x - y\big|^{d+ps}}\, dy = 0$$

$$\partial_t u + \int_{\mathbb{R}^d} \big(u(x) - u(y)\big) \left(\frac{\Phi\big(u(x) - u(y)\big)}{u(x) - u(y)} \frac{1}{|x - y|^{d+ps}} \right) dy = 0$$

Doubly nonlinear

$$\partial_t u + \int_{\mathbb{R}^d} \Phi\big[f\big(u(x)\big) - f\big(u(y)\big)\big]\mu\big(|x-y|\big)\,dy = 0.$$

Here f and Φ are non-decreasing functions and μ is a density of a Lévy measure with low singularity.

This is a nonlocal counterpart of the equation

$$\partial_t u = (\Delta_p)\big(u\,|u|^m\big), \quad \text{where} \quad \Delta_p v = \operatorname{div}\big(|\nabla v|^{p-2}\nabla v\big),$$

Non-local nonlinear "diffusion" equation

$$u_t + \mathcal{L}_u u = 0$$

with a nonlinear and nonlocal operator

$$\mathcal{L}_v u(x) = \int_{\mathbb{R}^d} \big[u(x) - u(y)\big] \rho\big(v(x), v(y); x, y\big)\, dy.$$

with a suitable *jump kernel* ρ.

Main property: Maximum principle. If

$$u(x_0) = \sup_{x \in \mathbb{R}^d} u(x) \geqslant 0$$

then

$$-\mathcal{L}_u u(x_0) \leqslant 0$$

(as long as everything is well-defined).

Jump kernel

$$\mathcal{L}_v u(x) = \int_{\mathbb{R}^d} \big[u(x) - u(y)\big]\rho\big(v(x), v(y); x, y\big)\, dy.$$

Suppose $\rho : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$

(A1) is a non-negative Borel function such that $\rho(a, b; x, y)$ is continuous with respect to a and b;

(A2) is symmetric

$$\rho(a, b; x, y) = \rho(b, a; y, x);$$

(A3) is monotonous in the following sense

$$(a-b)\rho(a, b; x, y) \geqslant (c-d)\rho(c, d; x, y) \quad \text{whenever } a \geqslant c \geqslant d \geqslant b;$$

(A4) we consider a homogeneous jump kernel

$$\rho(a, b; x, y) = \rho(a, b; |x - y|).$$

(A5) for every $R > 0$ there exists a function $m_R : [0, \infty) \to [0, \infty)$, which we call *$R$-majorant of ρ*, such that

$$\sup_{-R \leqslant a,b \leqslant R} \rho(a, b; x, y) \leqslant m_R(|x - y|)$$

$$\text{and } \int_{\mathbb{R}^d} (1 \wedge |y|)\, m_R(|y|)\, dy = K_R < \infty.$$

Jump kernel

$$\mathcal{L}_v u(x) = \int_{\mathbb{R}^d} \big[u(x) - u(y)\big]\rho\big(v(x), v(y); x, y\big)\, dy.$$

(A6) Regularity assumption.

The jump kernel ρ it is continuous with respect to the first two variables and it is locally Lipschitz-continuous outside diagonals, namely for every $\varepsilon > 0$ and every $R > \varepsilon$ there exists a constant $C_{R,\varepsilon}$ such that

$$\big|\rho(a, b, x, y) - \rho(c, d, x, y)\big| \leqslant C_{R,\varepsilon}\big(|a - c| + |b - d|\big)\, m_R\big(|x - y|\big)$$

for every $a, b, c, d \in [-R, R]$ such that $|a - b| \geqslant \varepsilon$ and $|c - d| \geqslant \varepsilon$.

Main result

Theorem (K., Kassmann, Krupski (2018))

Let ρ be a homogeneous jump kernel. For every initial datum $u_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$, the Cauchy problem has a unique (weak) solution

$$u \in L^\infty\big([0,\infty), L^1 \cap L^\infty \cap BV(\mathbb{R}^d)\big).$$

This solution has the following properties

- $\partial_t u \in L^\infty\big((0,\infty), L^1(\mathbb{R}^d)\big)$,
- *mass is conserved:* $\int_{\mathbb{R}^d} u(t,x)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx$ *for all* $t \geqslant 0$,
- L^p*-norm is non-increasing:* $\|u(t)\|_p \leqslant \|u_0\|_p$ *for all* $p \in [1,\infty]$ *and* $t \geqslant 0$,
- *if* u_0 *is nonnegative then the corresponding weak solution is nonnegative for almost all* $x \in \mathbb{R}^d$ *and* $t \geqslant 0$.

Moreover, for two solutions u and $\widetilde{u}$ corresponding to initial conditions u_0 and $\widetilde{u}_0$, respectively, we have

$$\|u(t) - \widetilde{u}(t)\|_1 \leqslant \|u_0 - \widetilde{u}_0\|_1 \qquad \text{for all} \quad t \geqslant 0.$$

First look

Lemma (Mass conservation)

Solution u satisfies $\int_{\mathbb{R}^d} u(t)\,dx = \int_{\mathbb{R}^d} u_0\,dx$ for every $t \geqslant 0$.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t)\,dx = \int_{\mathbb{R}^d} \frac{\mathrm{d}}{\mathrm{d}t} u(t)\,dx = -\int_{\mathbb{R}^d} \mathcal{L}_{u(t)} u(t)\,dx$$

$$= \iint_{\mathbb{R}^{2d}} \big[u(t,y) - u(t,x)\big]\rho\big(u(t,x), u(t,y); x, y\big)\,dy\,dx$$

$$= \iint_{\mathbb{R}^{2d}} \big[u(t,x) - u(t,y)\big]\rho\big(u(t,x), u(t,y); x, y\big)\,dx\,dy = 0.$$

We used the symmetry of the jump kernel ρ (property A2). □

Contraction in L^1

Theorem (Kato-type inequality)

If $u, v \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ are such that $\mathcal{L}_u u, \mathcal{L}_v v \in L^1(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}^d} (\mathcal{L}_u u - \mathcal{L}_v v)\, \mathrm{sgn}(u - v) \geqslant 0.$$

Proof.
Let $\rho\big(u(x), u(y); x, y\big) = \rho_{u,x,y}$.

$$\int_{\mathbb{R}^d} \big((\mathcal{L}_u u)(x) - (\mathcal{L}_v v)(x)\big)\, \mathrm{sgn}\,\big(u(x) - v(x)\big)\, dx$$

$$= \iint_{\mathbb{R}^{2d}} \big[u(x) - u(y)\big]\rho_{u,x,y} - \big[v(x) - v(y)\big]\rho_{v,x,y}\, dy\ \mathrm{sgn}\,\big(u(x) - v(x)\big)\, dx$$

$$= \frac{1}{2} \iint_{\mathbb{R}^{2d}} \big[u(x) - u(y)\big]\rho_{u,x,y} - \big[v(x) - v(y)\big]\rho_{v,x,y} \Big[\mathrm{sgn}\,\big(u(x) - v(x)\big) - \mathrm{sgn}\,\big(u(y) - v(y)\big)\Big]\, dy\, dx.$$

Contraction in L^1

Lemma

If u, v are weak solutions with initial data u_0 and v_0, respectively, such that

$$u, v \in W^{1,1}_{loc}([0,\infty), L^1(\mathbb{R}^d)) \quad \text{and} \quad \mathcal{L}_u u, \mathcal{L}_v v \in L^1_{loc}([0,\infty), L^1(\mathbb{R}^d))$$

then

$$\|u(t) - v(t)\|_1 \leqslant \|u_0 - v_0\|_1$$

for every $t \geqslant 0$.

Proof.

Multiply by $\operatorname{sgn}(u - v)$ and integrate over $\mathbb{R}^d$ the equality

$$\partial_t(u - v) = \mathcal{L}_u u - \mathcal{L}_v v.$$

In fact, we should begin with the relation

$$\int_0^\infty \int_{\mathbb{R}^d} \Big(\partial_t\big(u(t,x) - v(t,x)\big) + \mathcal{L}_{u(t)} u(t,x) - \mathcal{L}_{v(t)} v(t,x)\Big)\psi(t,x)\, dx\, dt = 0$$

□

The BV space (bounded variation)

Let $u \in L^1(\mathbb{R}^d)$ and suppose for $i = 1, \ldots, d$ there exist finite signed Radon measures λ_i such that

$$\int_{\mathbb{R}^d} u \, \partial_{x_i} \phi \, dx = -\int_{\mathbb{R}^d} \phi \, d\lambda_i \quad \text{for every } \phi \in C_c^\infty(\mathbb{R}^d).$$

We define

$$|Du|(\mathbb{R}^d) = \sup \Big\{ \sum_{i=1}^{d} \int_{\mathbb{R}^d} \Phi_i \, d\lambda_i : \Phi \in C_0(\mathbb{R}^d, \mathbb{R}^d), \|\Phi\|_{C_0(\mathbb{R}^d, \mathbb{R}^d)} < 1 \Big\}.$$

Then we say $u \in BV(\mathbb{R}^d)$ if $\|u\|_{BV} = 2\|u\|_1 + |Du|(\mathbb{R}^d) \leqslant \infty$.

Properties of $\mathcal{L}$

Lemma

For every $v \in L^\infty(\mathbb{R}^d)$ such that $\|v\|_\infty \leqslant R$ the linear operator $\mathcal{L}_v : BV(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ satisfies

$$\|\mathcal{L}_v u\|_1 \leqslant K_R \|u\|_{BV}.$$

Proof.

$$\begin{aligned}
\|\mathcal{L}_v u\|_1 &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \big[u(x) - u(y)\big] \rho_{v,x,y}\, dy \right| dx \\
&\leqslant \iint_{\mathbb{R}^{2d}} \frac{|u(x) - u(x-y)|}{1 \wedge |y|} (1 \wedge |y|) m_R(|y|)\, dx\, dy \\
&\leqslant \|u\|_{BV} \int_{\mathbb{R}^d} (1 \wedge |y|) m_R(|y|)\, dy = \|u\|_{BV} K_R.
\end{aligned}$$

□

Existence of solutions for regular jump kernels

Theorem

If ρ is a regular jump kernel then there exists a global classical solution to the Cauchy problem

$$u_t + \mathcal{L}_u u = 0, \qquad u(0) = u_0.$$

Proof.

$$F(u) = -\mathcal{L}_u u \quad \text{and} \quad \mathfrak{F}v(t) = u_0 + \int_0^t F(v(s))\, ds.$$

For $T > 0$ sufficiently small it is a contraction on $C\big([0,T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\big)$. Here, the equation $\partial_t u = F(u) = -\mathcal{L}_u u$ is satisfied in the usual sense. It can be shown that

$$\|u(t)\|_p \leqslant \|u_0\|_p \quad \text{for each} \quad p \in [1, \infty].$$

Thus the local classical solution may be extended to all $t \in [0, \infty)$ by a usual continuation argument. $\square$

Convergence of a sequence of approximations

Lemma

Let ρ be a homogeneous jump kernel and $u_0 \in BV(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.For each $\varepsilon > 0$ consider the unique solution u^ε of the initial value problem

$$\partial_t u^\varepsilon + \int_{\mathbb{R}^d} \left[u^\varepsilon(x) - u^\varepsilon(y)\right] \rho^\varepsilon\big((u^\varepsilon(x), u^\varepsilon(y); x, y)\big)\, dy = 0, \quad u^\varepsilon(0,\cdot) = u_0.$$

Then there exists a sequence $\varepsilon_j \to 0$ and a function $u \in C\big([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}^d)\big)$ such that

$$\lim_{j\to\infty} u^{\varepsilon_j}(t,x) = u(t,x) \quad \textit{a.e. in } [0,\infty) \times \mathbb{R}^d$$

Proof.

Let $v_\xi(x) = v(x+\xi)$. We have

$$(\mathcal{L}_v v)(x+\xi) = \int_{\mathbb{R}^d} \big(v(x+\xi) - v(y)\big)\rho_{v,x+\xi,y}\,dy$$

$$= \int_{\mathbb{R}^d} \big(v_\xi(x) - v_\xi(y)\big)\rho_{v_\xi,x,y}\,dy = \big(\mathcal{L}_{v_\xi} v_\xi\big)(x).$$

The function $\tilde{u} = u^{\varepsilon_j}$ is a solution with the initial condition $u_{\xi,0}(x) = u_0(x+\xi)$. Because of the L^1-contraction we have

$$\|\tilde{u}_\xi(t) - \tilde{u}(t)\|_1 \leqslant \|u_{\xi,0} - u_0\|_1$$

which give us

$$\|u(t)\|_{BV} \leqslant \|u_0\|_{BV}.$$

Hence $\mathcal{L}_{u(t)}u(t) \in L^1(\mathbb{R}^d)$ for every $t \geqslant 0$. Moreover,

$$\|\partial_t u\|_1 = \|\mathcal{L}_u u\|_1 \leqslant \|u(t)\|_{BV} K_R \leqslant \|u_0\|_{BV} K_R.$$

We conclude by applying Rellich-Kondrachov and Aubin-Lions-Simon lemmas.
□

Existence and uniqueness of solutions

Theorem

If ρ is a homogeneous jump kernel then there exists a unique weak solution $u \in L^\infty\big([0,\infty), BV \cap L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\big)$ to the problem for every $u_0 \in BV \cap L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Proof.

The approximating sequence of solutions $\{u_j^\varepsilon\}$ converges to $u(t,x)$.
By the Lebesgue convergence theorem, it is a weak solution.
Uniqueneness follows from the L^1-contraction property.

□

Comparison principle

Lemma (Comparison principle)

If u and $\widetilde{u}$ are weak solutions corresponding to initial conditions $u_0, \widetilde{u}_0 \in BV \cap L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, respectively, satisfying

$$u_0(x) \leqslant \widetilde{u}_0(x) \quad a.e \text{ in } x \in \mathbb{R}^d$$

then

$$u(t,x) \leqslant \widetilde{u}(t,x) \quad a.e. \text{ in } (t,x) \in [0,\infty) \times \mathbb{R}^d$$

Proof.

Using the L^1-contraction and the conservation of mass we get

$$\int_{\mathbb{R}^d} (u(t) - \widetilde{u}(t))^+ \, dx = \int_{\mathbb{R}^d} \frac{|u(t) - \widetilde{u}(t)| + u(t) - \widetilde{u}(t)}{2} \, dx$$

$$\leqslant \int_{\mathbb{R}^d} \frac{|u_0 - \widetilde{u}_0| + u_0 - \widetilde{u}_0}{2} \, dx = \int_{\mathbb{R}^d} (u_0 - \widetilde{u}_0)^+ \, dx$$

In particular, the equality $(u_0 - \widetilde{u}_0)^+ = 0$ a.e. implies $(u - \widetilde{u})^+ = 0$ a.e. □

EXAMPLES

Old examples: fractional porous medium equation

$$\Big((-\Delta)^s f(u)\Big)(x) = \int_{\mathbb{R}^d} \frac{f(u(x)) - f(u(y))}{|x-y|^{d+2s}}\, dy$$

$$= \int_{\mathbb{R}^d} \big(u(x) - u(y)\big) \left(\frac{f(u(x)) - f(u(y))}{u(x) - u(y)} \mu(|x-y|) \right) dy$$

Old examples: fractional porous medium equation

Lemma

Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be a non-decreasing function and

$$F(a,b) = \frac{f(a) - f(b)}{a - b}.$$

Then $\rho = F(a,b) \times \mu(|x-y|)$ is a jump kernel if

$$\int_{\mathbb{R}^d} \big(1 \wedge |y|\big)\mu(|y|)\, dy < \infty.$$

Proof.

For $a \geqslant c \geqslant d \geqslant b$ we have $f(a) \geqslant f(c) \geqslant f(d) \geqslant f(b)$, hence

$$(a-b)\frac{f(a)-f(b)}{a-b} = f(a) - f(b) \geqslant f(c) - f(d) = (c-d)\frac{f(c)-f(d)}{c-d}.$$

□

Old examples: fractional p-Laplacian

$$\partial_t u + \int_{\mathbb{R}^d} \frac{\Phi\big(u(x) - u(y)\big)}{|x-y|^{d+ps}}\, dy = 0, \quad \Phi(z) = z|z|^{p-2}.$$

Lemma

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a continuous, non-decreasing function satisfying $\Phi(x) \geqslant 0$ for $x \geqslant 0$, $\Phi(-x) = -\Phi(x)$ and $\lim_{x\to 0} \frac{\Phi(x)}{x} < \infty$, and let

$$F(a,b) = \frac{\Phi(a-b)}{a-b}$$

Then $\rho = F(a,b) \times \mu(|x-y|)$ is a jump kernel if $\int_{\mathbb{R}^d} \big(1 \wedge |y|\big)\mu(|y|)\, dy < \infty$.

Proof.

For $a \geqslant c \geqslant d \geqslant b$ and by using the fact that f is non-decreasing, we have

$$(a-b)\frac{\Phi(a-b)}{a-b} = \Phi(a-b) \geqslant \Phi(c-d) = (c-d)\frac{\Phi(c-d)}{c-d},$$

from which A3 follows. □

New examples: Wrocław type jump kernel

$$\partial_t u + \int_{\mathbb{R}^d} \big(u(x) - u(y)\big)\Big[\Big(f\,(u(x)) + f\,(u(y))\Big)\mu(|x-y|)\Big]\,dy$$

Lemma

Let $f : \mathbb{R} \to [0, \infty)$ be a continuous, convex function and

$$F(a, b) = f(a) + f(b)$$

Then $\rho = F(a, b) \times \mu(|x-y|)$ is a jump kernel if

$$\int_{\mathbb{R}^d} \big(1 \wedge |y|\big)\mu(|y|)\,dy < \infty.$$

New examples: Wrocław type jump kernel

Proof.

Let $a \geqslant c \geqslant d \geqslant b$ and $t \in [0,1]$ such that $d = ta + (1-t)b$. Then

$$\begin{aligned} f(a) + f(b) &\geqslant (1-t^2)f(a) + (1-t)^2 f(b) \\ &= (1-t)\big((1+t)f(a) + (1-t)f(b)\big) = (1-t)\big(f(a) + tf(a) + (1-t)f(b)\big). \end{aligned}$$

Because we assume f to be convex, we have

$$tf(a) + (1-t)f(b) \geqslant f\big(ta + (1-t)b\big) = f(d),$$

hence

$$f(a) + f(b) \geqslant (1-t)\big(f(a) + f(d)\big).$$

In the same fashion, by taking $s \in [0,1]$ such that $c = (1-s)a + sd$, we can show that

$$f(a) + f(d) \geqslant (1-s)\big(f(c) + f(d)\big).$$

Also

$$(1-t)(a-b) = a - \big(ta + (1-t)b\big) = a - d$$

and $(1-s)(a-d) = (c-d)$, so

$$(a-b)\big(f(a) + f(b)\big) \geqslant (1-s)(1-t)(a-b)\big(f(c) + f(d)\big) = (c-d)\big(f(c) + f(d)\big).$$

□

New examples: Bielefeld type jump kernel

$$\partial_t u - \int_{\mathbb{R}^d} \frac{u(y) - u(x)}{|y - x|^{d + \frac{1}{2} - \frac{1}{4} \sin \frac{1}{|x-y|}}} = 0$$

$$\partial_t u - \int_{B_1(x)} \frac{u(y) - u(x)}{|x - y|^{d + \frac{1}{2} - \frac{1}{4} \exp(-|u(y) - u(x)|)}} \, dy = 0$$

New examples: Bielefeld type jump kernel

Lemma

Let

$$f(a;x) = f_1(a)\mathbb{1}_{|x|<1}(x) + f_2(a)\mathbb{1}_{|x|\geqslant 1}(x),$$

where f_1 is non-decreasing, f_2 is non-increasing, f_1, f_2 are continuous and

$$0 < A_1 \leqslant f(a;x) \leqslant A_2 < 1.$$

Then

$$\rho(a,b;x,y) = |x-y|^{-d-f(a-b;x-y)}$$

is a jump kernel.

New examples: Bielefeld type jump kernel

Proof.

The conditions A1 and A2 are easy. We notice that for $a \geqslant c$ we have

$$a|x|^{-f_1(a)\mathbb{1}_{|x|<1}(x)} \geqslant c|x|^{-f_1(c)\mathbb{1}_{|x|<1}(x)},$$

because f_1 is non-decreasing and

$$a|x|^{-f_2(a)\mathbb{1}_{|x|\geqslant 1}(x)} \geqslant c|x|^{-f_2(c)\mathbb{1}_{|x|\geqslant 1}(x)},$$

because f_2 is non-increasing. Hence for $a \geqslant c \geqslant d \geqslant b$ we have $(a-b)\rho(a,b;x,y) \geqslant (c-d)\rho(c,d;x,y)$. This verifies A3. Finally,

$$\int_{\mathbb{R}^d} \big(1 \wedge |x-y|\big)\rho_{u,x,y}\,dy = \int_{\mathbb{R}^d} \big(1 \wedge |y|\big)|y|^{-d-f(u(x)-u(x-y);y)}\,dy$$

$$\leqslant \int_{|y|<1} |y|^{1-d-A_1}\,dy + \int_{|y|\geqslant 1} |y|^{-d-A_2}\,dy = \int_0^1 r^{-A_1}\,dr + \int_1^\infty r^{-1-A_2}\,dr,$$

therefore A4 is also satisfied. □

SEVERAL OTHER EXAMPLES

Lemma

Let ρ_1 and ρ_2 be jump kernels. Then

$$\rho = \alpha\rho_1 + \beta\rho_2$$

is a jump kernel for every $\alpha, \beta \geqslant 0$ (i.e. the set of jump kernels is a convex cone).

$$u_t + \mathcal{L}_u u = 0$$

where

$$\mathcal{L}_u u(x) = \int\limits_{\mathbb{R}^d} \big[u(x) - u(y)\big]\, \rho\big(u(x), u(y); x, y\big)\, dy.$$