Subcritical Galton-Watson branching processes with immigration in random environment

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Outline

This is ongoing joint work with Bojan Basrak (Zagreb).

An RWRE model

RWRE model by Kozlov and Solomon:

- $\{\alpha_i\}_{i\in\mathbb{Z}}$ iid random variables with values in $[0,1]$
- $\mathcal{A} = \sigma(\alpha_i : i \in \mathbb{Z})$ generated σ-algebra
- $X_0 = 0$ and

$$\mathbf{P}(X_{n+1} = X_n + 1 | \mathcal{A}, X_0, \ldots, X_n) = \alpha_i \ \text{ on } \{X_n = i\}$$
$$\mathbf{P}(X_{n+1} = X_n - 1 | \mathcal{A}, X_0, \ldots, X_n) = 1 - \alpha_i \ \text{ on } \{X_n = i\}$$

- X_n is not a Markov process
- $X_n \to \infty$ a.s., but $X_n/n \to 0$ a.s.

KKS result

Let $T_n = \min\{k : X_k = n\}$ = first hitting time of n.
Assume

$$\mathbf{E} \log \frac{1-\alpha}{\alpha} < 0, \text{ (positive drift)}$$

$$\mathbf{E} \left(\frac{1-\alpha}{\alpha} \right)^{\kappa} = 1, \text{ (Cramér's condition)}$$

$$\mathbf{E} \left(\frac{1-\alpha}{\alpha} \right)^{\kappa} \log_{+} \frac{1-\alpha}{\alpha} < \infty, \ \kappa > 0,$$

$\log \frac{1-\alpha}{\alpha}$ is non-arithmetic (not concentrated on $\delta\mathbb{Z}$ for any δ).
These are the assumption in Goldie's implicit renewal theorem.

KKS result

Theorem (Kesten & Kozlov & Spitzer (1975))

Then, for $\kappa \in (0, 2)$,

$$n^{-1/\kappa}(T_n - A_n) \xrightarrow{\mathcal{D}} \kappa - \text{stable rv.}$$

where $A_n \equiv 0$ for $\kappa < 1$, $A_n = nc_1$ for $\kappa > 1$.
For $\kappa > 2$

$$n^{-1/2}(T_n - nc) \xrightarrow{\mathcal{D}} N(0, 1).$$

Moreover, $n^{-\kappa}(X_n - B_n)$ also converges.

Branching connection

- $U_i^n =$ number of steps before T_n from i to $i-1$; $-\infty < i \leq n-1$.
- $T_n = n + 2\sum_{i \leq n-1} U_i^n$.
- It is enough to handle $\sum_{i=1}^{n} U_i^n$.

Branching connection

- U_i^n = number of steps before T_n from i to $i-1$; $-\infty < i \le n-1$.
- U_j^n given $\mathcal{A}, U_{j+1}^n, \dots, U_{n-1}^n$ is the sum of $U_{j+1}^n + 1$ iid random variables with joint distribution

$$\mathbf{P}(V = k) = \alpha_j (1-\alpha_j)^k, \quad k = 0, 1, \dots.$$

- U is a GW branching process with random offspring and immigration distribution. Given the environment α both the offspring and the immigration distribution is geometric with parameter α.

Subcritical GWI

Let $X_0 = 0$, and

$$X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1} =: \theta_{n+1} \circ X_n + B_{n+1}, \quad n \geq 0,$$

where the offsprings $\{A_i^{(n)} : i = 1, 2, \ldots, n = 1, 2, \ldots\}$ are iid, and independently, $\{B_n : n = 1, 2, \ldots\}$ iid.
Subcritical: $\mathbf{E}A < 1$.

Stationary distribution - existence

Theorem (Quine (1970), Foster & Williamson (1971))

If $m = \mathbf{E}A < 1$ *and* $\mathbf{E}\log B < \infty$ *then there exists a unique stationary distribution in the form*

$$X_\infty = B_1 + \theta_1 \circ B_2 + \theta_1 \circ \theta_2 \circ B_3 + \ldots = \sum_{i=0}^{\infty} \Pi_i \circ B_{i+1}.$$

Stationary distribution - tails

Theorem (Basrak & Kulik & Palmowski (2013))

(i) *If* $m = \mathbf{E}A < 1$, $\mathbf{E}A^2 < \infty$ *and* $\mathbf{P}(B > x)$ *is regularly varying with index* $-\alpha \in (-2, 0)$, *then*

$$\mathbf{P}(X_\infty > x) \sim c\,\mathbf{P}(B > x), \qquad c > 0.$$

(ii) *If* $m = \mathbf{E}A < 1$, *and* $\mathbf{P}(A > x)$ *is regularly varying with index* $\alpha \in (-2, -1)$, *and* $\mathbf{P}(B > x) \sim c'\mathbf{P}(A > x)$, $c' \geq 0$ *then*

$$\mathbf{P}(X_\infty > x) \sim c\,\mathbf{P}(A > x), \qquad c > 0.$$

GWIRE process - notation

- Δ the space of probability measures on $\mathbb{N} = \{0, 1, \ldots\}$
- $(\epsilon, \varpi), (\epsilon_1, \varpi_1), (\epsilon_2, \varpi_2), \ldots$ iid random elements in Δ^2 (the environment)
- $X_0 = 0$, and

$$X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1} =: \theta_{n+1} \circ X_n + B_{n+1}, \quad n \geq 0,$$

where, conditioned on the environments $\mathcal{E}, \mathcal{I}$, the variables $\{A_i^{(n)}, B_n : \, i = 1, 2, \ldots, n = 1, 2, \ldots\}$ are independent, for n fixed $(A_i^{(n)})_{i=1,2,\ldots}$ are iid with distribution ϵ_n, and B_n has distribution ϖ_n.

GWIRE process - notation

- Δ the space of probability measures on $\mathbb{N} = \{0, 1, \ldots\}$
- $X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1} =: \theta_{n+1} \circ X_n + B_{n+1}$
- $m(\delta) = \sum_{i=1}^{\infty} i\delta(\{i\})$ for $\delta \in \Delta$.
- Subcritical branching: $\mathbf{E} \log m(\epsilon) < 0$.

Existence of the stationary distribution

Theorem (Key (1987))

If $\mathbf{E}\log m(\epsilon) < 0$ *(offspring) and* $\mathbf{E}\log^+ m(\varpi) < \infty$ *(immigration) then a unique stationary distribution exists:*

$$X_\infty = B_1 + \theta_1 \circ B_2 + \theta_1 \circ \theta_2 \circ B_3 + \ldots = \sum_{i=0}^{\infty} \Pi_i \circ B_{i+1}.$$

Goldie's setup - the fixed point equation

$$X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1} =: \theta_{n+1} \circ X_n + B_{n+1}, \quad n \geq 0,$$

The stationary distribution satisfies the corresponding fixed point equation

$$X \stackrel{\mathcal{D}}{=} \sum_{i=1}^{X} A_i + B = \theta \circ X + B =: \Psi(X),$$

(θ, B) and X on the right-hand side are independent.

Assumptions

We are interested in the tail behavior, so need more assumption:

- Cramér's condition: $\mathbf{E}m(\epsilon)^\kappa = 1$ for some $\kappa > 0$.
- $\log m(\epsilon)$ is nonarithmetic (not concentrated on $\delta\mathbb{Z}$)
- $\mathbf{E}A^{\kappa\vee 2} < \infty$, (by Jensen: $\mathbf{E}m(\epsilon)^{\kappa\vee 2} < \infty$), and $\mathbf{E}B^\kappa < \infty$. For $\kappa > 1$ assume further that $\mathbf{E}A^{\kappa+\delta} < \infty$, $\mathbf{E}B^{\kappa+\delta} < \infty$ for some $\delta > 0$.

Main result

Theorem (Basrak & K (2019+))

Then

$$\mathbf{P}(X_\infty > x) \sim Cx^{-\kappa} \quad \text{as } x \to \infty,$$

where

$$C = \frac{1}{\kappa \mathbf{E} m(\epsilon)^\kappa \log m(\epsilon)} \mathbf{E}\left[\Psi(X_\infty)^\kappa - m(\epsilon)^\kappa X_\infty^\kappa\right] \geq 0.$$

Moreover, $C > 0$ *for* $\kappa \geq 1$*.*

Moments of the stationary distribution - deterministic

Lemma

Let X_n be a subcritical GWI
$X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1} =: \theta_{n+1} \circ X_n + B_n$,

$$X_\infty = B_1 + \theta_1 \circ B_2 + \theta_1 \circ \theta_2 \circ B_3 + \ldots = \sum_{i=0}^{\infty} \Pi_i \circ B_{i+1}.$$

$\mathbf{E} X_\infty^\alpha < \infty$ *whenever* $\mathbf{E} B^\alpha < \infty$ *and* $\mathbf{E} A^\alpha < \infty$, $\alpha \geq 1$.

- Quine (1970): $\alpha = 2$;
- Barczy, Nedényi, Pap (2018): $\alpha = 2, 3$.
- Szűcs (2014): under ergodicity conditions, general α
- K - Wiandt (2019+): multitype case

Proof

- Enough to show that $M_\alpha(n) := \mathbf{E}(\theta_1 \circ \theta_2 \circ \dots \theta_n \circ B_{n+1})^\alpha$ decreases exponentially.
- $\forall \lambda \in (\mu^\alpha, 1) \ \ \exists n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\mu^\alpha \mathbf{E}\left(\frac{\sum_{i=1}^n A_i}{n\mu}\right)^\alpha < \lambda.$$

- $C_0 = \mathbf{E}\left(\sum_{i=1}^{n_0} A_i\right)^\alpha$

Proof

- $\mu^\alpha \mathbf{E}\left(\frac{\sum_{i=1}^n A_i}{n\mu}\right)^\alpha < \lambda$ for $n \geq n_0$.
- $C_0 = \mathbf{E}\left(\sum_{i=1}^{n_0} A_i\right)^\alpha$
- Y independent of $A_1, A_2, \ldots$,

$$\begin{aligned}\mathbf{E}\left(\sum_{i=1}^{Y} A_i\right)^\alpha &= \sum_{n=1}^{\infty} \mathbf{P}(Y=n)\mathbf{E}\left(\sum_{i=1}^{n} A_i\right)^\alpha \\ &\leq \mathbf{P}(Y \geq 1)C_0 + \sum_{n=n_0}^{\infty} \mathbf{P}(Y=n)n^\alpha \lambda \\ &\leq C_0 \mathbf{E}Y + \lambda \mathbf{E}Y^\alpha.\end{aligned}$$

Proof

$$M_\alpha(n) := \mathbf{E}(\theta_1 \circ \theta_2 \circ \ldots \circ \theta_n \circ B_{n+1})^\alpha$$

$\mathbf{E}\left(\sum_{i=1}^{Y} A_i\right)^\alpha \leq C_0 \mathbf{E} Y + \lambda \mathbf{E} Y^\alpha$ implies the recursion

$$M_\alpha(n) \leq C_0 M_1(n-1) + \lambda M_\alpha(n-1).$$

Iterating the inequality, we obtain

$$M_\alpha(n) \leq C_0 \sum_{i=0}^{n-1} \lambda^i \mu^{n-i} + \lambda^n \mathbf{E} B^\alpha.$$

Moments of the stationary distribution - random

Lemma

Assume that $\mathbf{E}m(\epsilon)^\alpha < 1$, $\mathbf{E}A^\alpha < \infty$, *and* $\mathbf{E}B^\alpha < \infty$. *Then* $\mathbf{E}X_\infty^\alpha < \infty$.

By the conditional Jensen inequality $\mathbf{E}m(\epsilon)^t \leq \mathbf{E}A^t$ for $t \geq 1$, while for $t \leq 1$ $\mathbf{E}m(\epsilon)^t \geq \mathbf{E}A^t$.

Goldie's condition

$\Psi(n) = \sum_{i=1}^{n} A_i + B$. The fixed point equation

$$X \stackrel{\mathcal{D}}{=} \Psi(X),$$

Lemma

Assume $\mathbf{E} m(\epsilon)^\kappa = 1$, $\mathbf{E} A^{\kappa \vee 2} < \infty$, *(*$\mathbf{E} m(\epsilon)^{\kappa \vee 2} < \infty$*), and* $\mathbf{E} B^\kappa < \infty$. *For* $\kappa > 1$ *assume further that* $\mathbf{E} A^{\kappa+\delta} < \infty$, $\mathbf{E} B^{\kappa+\delta} < \infty$ *for some* $\delta > 0$. *Then*

$$\mathbf{E} \left| \Psi(X)^\kappa - (m(\epsilon) X)^\kappa \right| < \infty.$$

The setup

- $X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1} =: \theta_{n+1} \circ X_n + B_{n+1}$, $n \in \mathbb{Z}$, strictly stationary.
- Assume $\mathbf{P}(X_0 > x) \sim cx^{-\kappa}$, $c > 0$.
- Let a_n such that $n\mathbf{P}(X_0 > a_n) \sim 1$.

Tail process

Theorem

For any integers $k, \ell \geq 0$

$$\mathcal{L}\left(\frac{X_0}{x}, \frac{X_{-k}}{X_0}, \ldots, \frac{X_0}{X_0}, \ldots, \frac{X_\ell}{X_0} \,\middle|\, X_0 > x \right) \xrightarrow{d} (Y_0, \Theta_{-k}, \ldots, \Theta_0, \ldots, \Theta_\ell) \,.$$

where Θ_n *is a multiplicative random walk, independent of* Y_0*, and* $\mathbf{P}(Y_0 > u) = u^{-\kappa}$*,* $u \geq 1$*.*

Consequence of a general results by Janssen & Segers (2014).

Anticlustering and $\mathcal{A}'(a_n)$ conditions

Anticlustering: If $r_n = o(n)$, then for any $u > 0$

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbf{P}\left(\max_{k\le|t|\le r_n} X_t > a_n u | X_0 > a_n u \right) = 0.$$

Ergodicity implies ($\mathcal{A}'(a_n)$):

$$\mathbf{E}\exp\left\{ -\sum_{i=1}^{n} f\left(\frac{i}{n}, \frac{X_i}{a_n} \right) \right\} - \prod_{k=1}^{k_n} \mathbf{E}\exp\left\{ -\sum_{i=1}^{r_n} f\left(\frac{kr_n}{n}, \frac{X_i}{a_n} \right) \right\} \to 0.$$

Point process convergence

Anticlustering and $\mathcal{A}'(a_n)$ implies for each $u > 0$

$$\sum_{i=1}^{n} \delta_{(i/n, X_i/a_n)}\Big|_{[0,1]\times(\mathbb{R}\setminus[-u,u])} \xrightarrow{\mathcal{D}} N^{(u)}.$$