Brownian motion with general drift

Damir Kinzebulatov

UNIVERSITÉ LAVAL

joint with Yuliy A. Semënov, Toronto

Probability and Analysis 2019

The subject of this talk: existence and uniqueness (in law) of weak solution to

$$dX_t = -b(X_t)dt + dW_t, \quad X(0) = x \in \mathbb{R}^d,$$

for a locally unbounded general $b : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 3$

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R. Bass-Z.-Q. Chen (2003) for a larger class: $b \in \mathbf{K}_0^{d+1}$

A vector field b is in the Kato class $\mathbf{K}_\delta^{d+1}$, $0 < \delta < 1$, if

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$$\mathbf{K}_0^{d+1} \equiv \mathbf{K}^{d+1} := \bigcap_{\delta > 0} \mathbf{K}_\delta^{d+1}$$

There are $b \in \mathbf{K}_0^{d+1}$ such that $|b| \notin L^{1+\varepsilon}_{\rm loc}$ (this excludes Girsanov transform)

The role of Kato class:

Qi S. Zhang (1996): Gaussian bounds on the heat kernel of $-\Delta + b \cdot \nabla$, $\mathbf{K}_0^{d+1}$

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R. Bass-Z.-Q. Chen (2003): SDEs with $b \in \mathbf{K}_0^{d+1}$ – **This talk: larger class of drifts** b

A vector field b is in the class $\mathbf{F}_\delta^{\frac{1}{2}}$, $0<\delta<1$, if

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Larger than Kato class:

$$\mathbf{K}_\delta^{d+1}\subsetneq\mathbf{F}_\delta^{\frac{1}{2}}$$

e.g. by interpolation between $\||b|(\lambda-\Delta)^{-\frac{1}{2}}\|_{1\to 1}\leq\delta$ and (by duality) $\|(\lambda-\Delta)^{-\frac{1}{2}}|b|\|_\infty\leq\delta$

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Remark: $b \in \mathbf{F}_\delta^{\frac{1}{2}}$ ensures that the form-sum $(\lambda - \Delta)^{\frac{1}{2}} \dot{+} |b|$ is well defined

Theorem[1]

Let $b \in \mathbf{F}_\delta^{1/2}$ with $0 < \delta < m_d^{-1} 4 \frac{d-2}{(d-1)^2}$, where $m_d := \pi^{\frac{1}{2}} (2e)^{-\frac{1}{2}} d^{\frac{d}{2}} (d-1)^{\frac{1-d}{2}}$.

[1] D. Kinzebulatov, Yu.A. Semënov, arXiv:1710.06729

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There is Feller generator $\Lambda \supset -\Delta + b \cdot \nabla$ on
$C_\infty = \{f \in C_b : \lim_{x\to\infty} f(x) = 0\}$ (sup-norm)[2]

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Let $\mathbb{P}_x$ be the probability measures on $D([0,\infty), \bar{\mathbb{R}}^d)$ determined by $e^{-t\Lambda_\infty(b)}$.

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Then:

1. $\mathbb{P}_x$ are concentrated on $C([0,\infty), \mathbb{R}^d)$
2. $\mathbb{E}_{\mathbb{P}_x} \int_0^t |b(X(s))| ds < \infty$ and there exists a d-dimensional Brownian motion W_t such that $\mathbb{P}_x$ a.s.

$$X_t = x - \int_0^t b(X_s) ds + \sqrt{2} W_t, \quad t \geq 0$$

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3. (Uniqueness) If $\{\mathbb{Q}_x\}_{x \in \mathbb{R}^d}$ is another weak solution such that

$$\mathbb{Q}_x = w\text{-}\lim_n \mathbb{P}_x(\tilde{b}_n) \quad \text{for every } x \in \mathbb{R}^d,$$

where $\{\tilde{b}_n\} \subset \mathbf{F}_\delta^{1/2} \cap C^\infty$ then $\{\mathbb{Q}_x\}_{x \in \mathbb{R}^d} = \{\mathbb{P}_x\}_{x \in \mathbb{R}^d}$.

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Let $b(x) = c|x|^{-2}x$ (is in $\mathbf{F}_\delta^{\frac{1}{2}}$)

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(b has a critical-order singularity at $x = 0$, i.e. SDE "senses" c)

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For example, for critical drift $b(x) = \delta \frac{d-2}{2}|x|^{-2}x$, via desingularizing weights of Milman-Semënov:

Metafune, Sobajima, Spina, 2017 established sharp two-sided estimates

$$e^{-t\Lambda}(x,y) \approx e^{t\Delta}(x,y)\varphi_t(y),$$

where

$$\varphi_t(y) = \begin{cases} |y|^{-\sigma}, & \frac{|y|}{\sqrt{t}} \leq 1, \\ \frac{1}{2}, & \frac{|y|}{\sqrt{t}} \geq 2, \end{cases}$$

with $\sigma = \frac{d-2}{2}(1 - \sqrt{1-\delta})$

Suppose $b \in C_b$

Then $\Lambda = -\Delta + b \cdot \nabla$, $D(\Lambda) = (1-\Delta)^{-1} C_\infty$ is a Feller generator

By a classical result, $e^{-t\Lambda}$ determines probability measures $\mathbb{P}_x$ on $D([0,\infty), \bar{\mathbb{R}}^d)$ such that

$$t \mapsto f(X_t) - f(x) - \int_0^t (\Lambda f)(X_s)ds \text{ is a } \mathbb{P}_x \text{ martingale}$$

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Here: know everything about $D(\Lambda)$

Let $b \in \mathbf{F}_\delta^{\frac{1}{2}}$

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Implicit connection: Set $b_n = e^{\varepsilon_n \Delta}(\mathbf{1}_n b)$, $\mathbf{1}_n = \mathbf{1}_{\{|x| \leq n, |b(x)| \leq n\}}$, define

$$T_t := s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda(b_n)} \quad \text{(existence?)}$$

where $\Lambda(b_n) = -\Delta + b_n \cdot \nabla$, $D(\Lambda) = (1-\Delta)^{-1} C_\infty$

The generator of $T_t =: e^{-t\Lambda}$ is a realization of $-\Delta + b \cdot \nabla$ on C_∞

$D(\Lambda) = ?$ (can't hope to have an exhaustive description; $C_c^\infty \not\subset D(\Lambda)$)

A simpler problem: $-\Delta + b \cdot \nabla$ as generator on some L^p

The Kato class $\mathbf{K}_\delta^{d+1}$ condition says that $b \cdot \nabla$ is a Miyadera perturbation of $-\Delta$ in L^1, so

$$\Lambda_1 := -\Delta + b \cdot \nabla, \quad D(\Lambda_1) = D(-\Delta) \text{ in } L^1$$

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Moreover, since $\|e^{-t\Lambda_1} f\|_\infty \leq \|f\|_\infty$, $f \in L^1 \cap L^\infty$, one can define

$$e^{-t\Lambda_p} := \left[e^{-t\Lambda_1} \upharpoonright_{L^1 \cap L^p} \right]_{L^p \to L^p}^{\text{clos}}, \qquad 1 \leq p < \infty$$

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If $b \in \mathbf{F}_\delta^{\frac{1}{2}}$: The standard tools of Perturbation Theory (such as Miyadera's and Hille's theorems, form-method) **are not applicable**

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e.g. form-method

$$\langle \nabla u, \nabla u \rangle + \langle b \cdot \nabla u, u \rangle \geq (1-\varepsilon)\langle \nabla u, \nabla u \rangle - \frac{1}{4\varepsilon}\langle |b|^2 u, u \rangle, \quad \varepsilon < 1$$

(but b is only in $L^1_{\rm loc}$!)

Well, not quite:

We can construct $\Lambda_2 \supset -\Delta + b \cdot \nabla$ with $b \in \mathbf{F}_\delta^{\frac{1}{2}}$ using ideas of Lions and Hille[3]

(think of $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ but with 6 spaces)

Interesting: In comparison with the Kato-Lions-Lax-Milgram-Nelson (KLMN) Theorem, our approach yields a larger class of vector fields + greater regularity

[3] D. Kinzebulatov, Yu. A. Semenov, "On the theory of the Kolmogorov operator in the spaces L^p and C_∞. I", arXiv:1709.08598

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(contains e.g. Hardy drift $b(x) = \delta \frac{d-2}{2}|x|^{-2}x$ and $L^{d,\infty}$, Campanato-Morrey, **but not** $\mathbf{K}_0^{d+1}$) constructed Feller semigroup via Moser-type iterations

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$$\mathbf{F}_{\delta_1} + \mathbf{K}_{\delta_2}^{d+1} \subset \mathbf{F}_\delta^{\frac{1}{2}}, \quad \delta = \delta_1 + \delta_2$$

Start with an operator-valued function on $\mathrm{Re}\zeta \geq \lambda$

$$\Theta_p(\zeta, b) := (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{1}{2} - \frac{1}{2q}} Q_p (1 + T_p)^{-1} G_p (\zeta - \Delta)^{-\frac{1}{2r'}}$$

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for $r < p < q < \infty$ (a "candidate for the resolvent"; formally, Neumann series)

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(via L^p-inequalities of Liskevich-Semënov (1996) between $(\lambda - \Delta)^{\frac{1}{2}}$ and "potential" $|b|$)

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$\Rightarrow \|T_p\|_{p\to p} < 1$ (via $|\nabla(\zeta-\Delta)^{-1}(x,y)| \leqslant m_d(\kappa_d \mathrm{Re}\zeta - \Delta)^{-\frac{1}{2}}(x,y)$) iff

$$p \in \left]\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right[$$

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$b \in \mathbf{F}_\delta^{\frac{1}{2}} \Leftrightarrow \||b|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{2}}|b|^{\frac{1}{2}}\|_{2\to 2} \leq m_d\delta$ yields (**principal step**)

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(via L^p-inequalities of Liskevich-Semënov (1996) between $(\lambda - \Delta)^{\frac{1}{2}}$ and "potential" $|b|$)

$\Rightarrow \|T_p\|_{p\to p} < 1$ (via $|\nabla(\zeta - \Delta)^{-1}(x,y)| \leqslant m_d(\kappa_d \mathrm{Re}\zeta - \Delta)^{-\frac{1}{2}}(x,y)$) iff

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(Don't need any of this for the Kato class. For Kato $p \in [1,\infty[$)

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By the Trotter Approximation Theorem, the limit $\Theta_p(\zeta,b)$ is the resolvent of the generator Λ_p of a holomorphic semigroup on L^p $\square$

$\Lambda_p \supset -\Delta + b\cdot\nabla$

[4] apriori the domain of $\zeta \mapsto (\zeta+\Lambda_p(b_n))^{-1}$ may depend on n

Thus,

$$\Theta_p(\zeta, b) = (\zeta + \Lambda_p)^{-1}$$

$\Rightarrow$ The very definition of $\Theta_p(\zeta, b)$ yields

$$D(\Lambda_p) \subset \mathcal{W}^{1+\frac{1}{q},p}, \quad \text{any } q > p$$

for $p \in \left]\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right[$

For $p > d-1$, by the Sobolev Embedding Theorem

$$\Theta_p(\mu,b)L^p \subset C_\infty \cap C^{0,\gamma}$$

Define

$$(\mu + \Lambda(b))^{-1} := [\Theta_p(\mu,b) \restriction_{C_\infty \cap L^p}]^{\text{clos}}_{C_\infty}$$

(appeal again to Trotter but now in C_∞)

$\Lambda(b) \supset -\Delta + b \cdot \nabla$ is Feller generator[5]

Remarks:

We have transferred the proof of convergence in C_∞ to L^p, $p > d-1$, a space having much weaker topology (locally)

Earlier proofs, for $b \in \mathbf{K}_0^{d+1}$, verified convergence in C_∞ directly (Arzela-Ascoli)

[5] D. Kinzebulatov, Annali SNS, 2017

A comment about Schrödinger operators ...

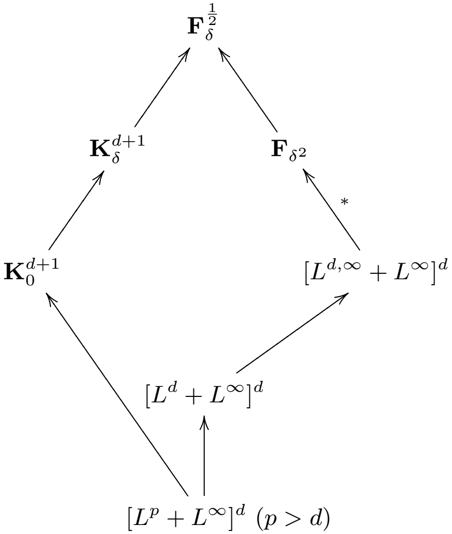
$-\Delta + V$ is a Feller generator if V is in the Kato class $\mathbf{K}_0^d$ (E.-M. Ouhabaz, P. Stollmann, K.-Th. Sturm, J. Voigt, 1994)

A comment about Schrödinger operators . . .

$-\Delta + V$ is a Feller generator if V is in the Kato class $\mathbf{K}_0^d$ (E.-M. Ouhabaz, P. Stollmann, K.-Th. Sturm, J. Voigt, 1994)

For potentials, can't go beyond the Kato class $\mathbf{K}_0^d$ (J. Voigt, 1995)

$-\Delta + b \cdot \nabla$ – classes of vector fields b studied in the literature



Remark: For Schrödinger operators there is no this dichotomy

To the SDEs[6] ...

[6] D. Kinzebulatov, Yu.A. Semënov, "Brownian motion with general drift", arXiv:1710.06729

Let $b \in \mathbf{F}_\delta^{1/2}$. We need:

1. Weight

$$\rho(x) := (1 + l|x|^2)^{-\nu} \quad \text{with } l, \nu > 0 \text{ to be chosen}$$

Technical Lemma

Fix $p > d - 1$. Then for $l > 0$ small,

$$\left\|\rho(\mu + \Lambda(b_n))^{-1} h\right\|_\infty \leq K_1 \|\rho h\|_p, \qquad (*)$$

$$\left\|\rho(\mu + \Lambda(b_n))^{-1} |b_m| h\right\|_\infty \leq K_2 \||b_m|^{\frac{1}{p}} \rho h\|_p, \quad h \in C_c^\infty \qquad (**)$$

Proof: Commute ρ in resolvent representation via $|\nabla \rho| \leq C_1 \sqrt{l} \rho$, $|\Delta \rho| \leq C_2 l \rho$

$(*)$ allows to control behaviour at ∞ (replaces upper Gaussian bound)
$(**)$ allows to control $\int_0^t \mathbb{E}_x[b_n \cdot \nabla g](X_s) ds$ (e.g. in martingale problem)

We need:

2.

$$e^{-t\Lambda(b)} = s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda(b_n)} \quad \text{(in fact, that's how we construct } e^{-t\Lambda(b)}\text{)},$$

where $\Lambda(b_n) = -\Delta + b_n \cdot \nabla$, $D(\Lambda(b_n)) = (1-\Delta)^{-1} C_\infty$

The approximation result and the Technical Lemma allow to prove:

1. Trajectories do not escape to ∞
2. $\mathbb{P}_x$ solve martingale problem on $D([0,\infty),\mathbb{R}^d)$

$$t \mapsto f(X_t) - f(x) - \int_0^t (-\Delta f + b \cdot \nabla f)(X_s)ds$$

for all $f \in C_c^\infty$ (not for $D(\Lambda(b))$ – don't even know if it separates closed sets)

3. A standard argument yields that the trajectories are continuous
4. SDE (use weights to get $f(x) = x_i$, $f(x) = x_i x_j$)
5. Uniqueness: The resolvent representation + a standard argument

Remark: Uniqueness in law? $\nabla(\lambda + \Lambda(b))^{-1} f \notin L^\infty$ if $b \in \mathbf{F}_\delta^{\frac{1}{2}}$

Recently: Itô SDE (also Stratonovich)

$$X(t) = x - \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s), \quad x \in \mathbb{R}^d,$$

1) $b \in \mathbf{F}_\delta$, i.e. $|b|^2 \in L^2_{\rm loc} \equiv L^2_{\rm loc}(\mathbb{R}^d)$ and

$$\||b|(\lambda - \Delta)^{-\frac{1}{2}}\|_{2\to 2} \leqslant \sqrt{\delta}$$

(previous examples, e.g. weak L^d, but not the Kato class)

[7] D.Kinzebulatov, Yu.A.Semenov, "Stochastic differential equations with singular (form-bounded) drift" arXiv:1904.01268

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2) $a := \sigma\sigma^\intercal$ is uniformly elliptic and $\nabla_r a_{\cdot\ell} \in \mathbf{F}_{\gamma_{r\ell}}$ $(1 \leq r, \ell \leq d)$.

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$$a(x) = I + c\frac{x \otimes x}{|x|^2}, \quad c > -1$$

$$a(x) = I + c(\sin\log(|x|))^2 e \otimes e, \quad e \in \mathbb{R}^d, |e| = 1,$$

or their infinite sum (geometry of discontinuities is not important)

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Proof[7]: A different technique (essentially, "differentiate the equation without differentiating the drift") – better control over relative bound δ

[7] D.Kinzebulatov, Yu.A.Semenov, "Stochastic differential equations with singular (form-bounded) drift" arXiv:1904.01268

Thank you

http://archimede.mat.ulaval.ca/pages/kinzebulatov

Liskevich-Semënov:

A, a symmetric Markov generator. Then $r \in]1,\infty[$

$$0 \leqslant u \in D(A_r) \Rightarrow v := u^{\frac{r}{2}} \in D(A^{\frac{1}{2}}) \text{ and } c_r^{-1}\|A^{\frac{1}{2}}v\|_2^2 \leqslant \langle A_r u, u^{r-1}\rangle.$$

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Let $A_r u = |f|,\ f \in L^r$. Note that $\|u\|_r \leqslant \mu^{-\frac{1}{2}} \|f\|_r$. Since $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, we have

$$(c_r \delta)^{-1} \||b|^{\frac{1}{2}} v\|_2^2 \leqslant \langle A_r u, u^{r-1} \rangle,$$

and so $\||b|^{\frac{1}{r}} u\|_r^r \leqslant c_r \delta \|f\|_r \|u\|_r^{r-1}$, $\||b|^{\frac{1}{r}} A_r^{-1} |f|\|_r^r \leqslant c_r \delta \mu^{-\frac{r-1}{2}} \|f\|_r^r$. $\square$

Let Y be a (complex) Banach space. A pseudo-resolvent R_ζ is a function defined on a subset $\mathcal{O}$ of the complex ζ-plane with values in $\mathcal{B}(Y)$ such that

$$R_\zeta - R_\eta = (\eta - \zeta) R_\zeta R_\eta, \quad \zeta, \eta \in \mathcal{O}.$$

Clearly, R_ζ have common null-set.

THEOREM (E. Hille)

If the null-set of R_ζ is $\{0\}$, then R_ζ is the resolvent of a closed linear operator A, the range of R_ζ coincides with $D(A)$, and $A = R_\zeta^{-1} - \zeta$.

Proof: Put $A := R_\zeta^{-1} - \zeta$. Since R_ζ is closed, so is R_ζ^{-1} and A. A straightforward calculation shows that $(\zeta + A) R_\zeta f = f$, $f \in Y$, and $R_\zeta(\zeta + A) g = g$, $g \in D(A)$, as needed.

THEOREM (E. Hille)

If there exists a sequence of numbers $\{\mu_k\} \subset \mathcal{O}$ such that $\lim_k |\mu_k| = \infty$ and $\sup_k \|\mu_k R_{\mu_k}\|_{Y \to Y} < \infty$, then the set $\{y \in Y : \lim_k \mu_k R_{\mu_k} y = y\}$ is contained in the closure of the range of R_ζ.

Proof: Indeed, let $\lim_k \mu_k R_{\mu_k} y = y$. That is, for every $\varepsilon > 0$, there exists k such that $\|y - \mu_k R_{\mu_k} y\| < \varepsilon$, so y belongs to the closure of the range of R_ζ.

Consider a sequence of C_0 semigroups e^{-tA_k} on a (complex) Banach space Y.

Theorem (H.F. Trotter)

Let

$$\sup_k \|(\mu + A_k)^{-1}\|_{Y \to Y} \leq \mu^{-1}, \quad \mu > \omega,$$

or

$$\sup_k \|(z + A_k)^{-1}\|_{Y \to Y} \leq C|z|^{-1}, \quad \operatorname{Re} z > \omega,$$

and let $s\text{-}\lim_{\mu \to \infty} \mu(\mu + A_k)^{-1} = 1$ *uniformly in* k*. Let* $s\text{-}\lim_k (\zeta + A_k)^{-1}$ *exist for some* ζ *with* $\operatorname{Re} \zeta > \omega$*. Then there is a* C_0 *semigroup* e^{-tA} *such that*

$$(z + A_k)^{-1} \xrightarrow{s} (z + A)^{-1} \quad \textit{for every } \operatorname{Re} z > \omega,$$

and

$$e^{-tA_k} \xrightarrow{s} e^{-tA}$$

uniformly in any finite interval of $t \geq 0$.

Theorem (Hille Pertrubation Theorem)

Let e^{-tA} be a symmetric Markov semigroup, K a linear operator in L^r for some $r \in]1,\infty[$. If for some $\lambda > 0$ $\|K(\lambda + A_r)^{-1}\|_{r \to r} < \frac{1}{2}$, then $-\Lambda_r := -A_r - K$ of domain $D(A_r)$ is the generator of a quasi bounded holomorphic semigroup on L^r.

$$m_d := \pi^{\frac{1}{2}} (2e)^{-\frac{1}{2}} d^{\frac{d}{2}} (d-1)^{\frac{1-d}{2}}$$