Hardy–Littlewood maximal operators in non-doubling setting

Dariusz Kosz

Wrocław University of Science and Technology
Faculty of Pure and Applied Mathematics

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$$\mathcal{M}^{\mathrm{c}} f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \, d\mu, \qquad x \in X,$$

and

$$\mathcal{M} f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_{B} |f| \, d\mu, \qquad x \in X.$$

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Definition

We say that $\mathfrak{X}$ is *doubling if there exists* $C > 0$ *such that*

$$\mu(B(x, 2r)) \leq C\, \mu(B(x, r))$$

holds for each $x \in X$ *and* $r > 0$.

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Question: What we can say about $\mathcal{M}^{\mathrm{c}}$ and $\mathcal{M}$ if $\mathfrak{X}$ is non-doubling?

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3. (H.-Q. Li, 2004) For any fixed $p_0 \in (1,\infty)$ there exists $\mathfrak{X}$ such that the associated operator $\mathcal{M}^{\mathrm{c}}$ is of strong type (p,p) if and only if $p > p_0$.

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Our goal: to extend the results of Li by using the appropriate class of non-doubling spaces.

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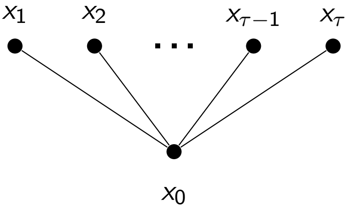

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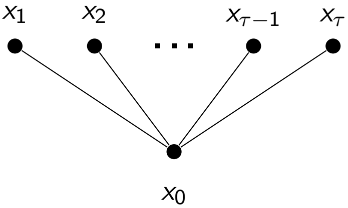

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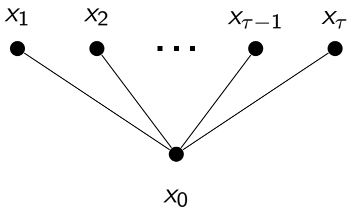

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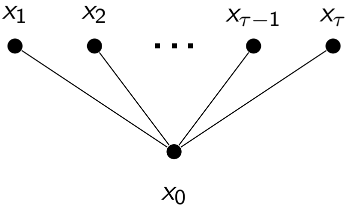

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Measure: $\mu(\{x_0\}) = 1$, $\mu(\{x_i\}) = m$, $m \geq 1$, for each $i \in \{1, \dots, \tau\}$.

Let $g = \chi_{\{x_0\}}$. Then for each $i \in \{1, \dots, \tau\}$ we have

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$$\|\mathcal{M}^{\mathrm{c}} g \cdot \chi_{S \setminus \{x_0\}}\|_p \simeq \frac{1}{m} \cdot (m\tau)^{1/p} = m^{-1+1/p} \tau^{1/p} \|g\|_p.$$

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In fact, one can show that

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If we take $m_n = n$ and $\tau_n = \lfloor n^{p_0 - 1} \rfloor$, $p_0 \in [1, \infty)$, then $m_n^{-1+1/p} \tau_n^{1/p} \simeq n^{-1+p_0/p} \to \infty$ if and only if $p \in [1, p_0)$.

Theorem SCT (Space Combining Technique)

For each $n \in \mathbb{N}$ *consider* $\mathfrak{Y}_n = (Y_n, \rho_n, \mu_n)$ *satisfying* $\mu_n(Y_n) < 2^{-n}$ *and* $\mathrm{diam}(Y_n) \leq 2$.

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Then for each $p \in [1, \infty)$ *we have*

$$\mathbf{C}_p(\mathfrak{Y}, \mathcal{M}^{\mathrm{c}}) \simeq \sup_{n \in \mathbb{N}} \mathbf{C}_p(\mathfrak{Y}_n, \mathcal{M}^{\mathrm{c}})$$

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Theorem 1

Fix $p_0 \in [1, \infty)$. There exists a non-doubling space such that the associated operators $\mathcal{M}^{\mathrm{c}}$ and $\mathcal{M}$ are of strong type (p, p) if and only if $p \in [p_0, \infty]$.

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2. Rescale the measures ($\mu_n' = d_n \cdot \mu_n$ for some $d_n > 0$) to obtain $\mu_n'(S_n) \leq 2^{-n}$.
3. Use Theorem SCT for $\mathfrak{Y}_n = (S_n, \rho_n, \mu_n')$, $n \in \mathbb{N}$.

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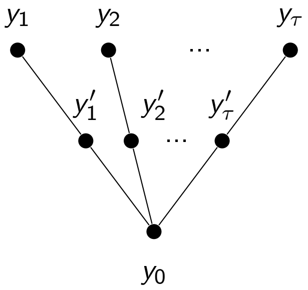

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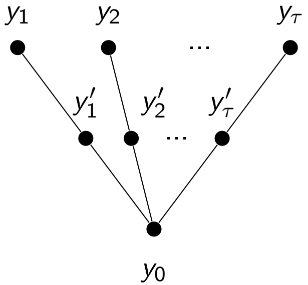

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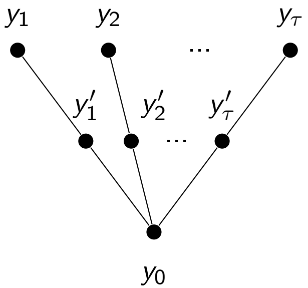

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Important balls: $B(y_i', \frac{3}{2}) = \{y_0, y_i', y_i\}$, $i \in \{1, \ldots, \tau\}$.

One can show that

$$\mathbf{C}_p(\mathcal{T}, \mathcal{M}^{\mathrm{c}}) \simeq 1$$

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Theorem 2

Fix $p_0 \in [1, \infty)$. There exists a non-doubling space such that $\mathcal{M}$ is of strong type (p, p) if and only if $p \in [p_0, \infty]$, while $\mathcal{M}^{\mathrm{c}}$ is of strong type $(1, 1)$.

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2. For each $n \in \{2, 4, 6, \dots\}$ let $\mathcal{T}_n$ be the space of type $\mathcal{T}$ with $m_n = n$ and $\tau_n = \lfloor n^{p_2 - 1} \rfloor$.

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Proof.

1. For each $n \in \{1, 3, 5, \dots\}$ let $\mathfrak{S}_n$ be the space of type $\mathfrak{S}$ with $m_n = n$ and $\tau_n = \lfloor n^{p_1 - 1} \rfloor$.
2. For each $n \in \{2, 4, 6, \dots\}$ let $\mathcal{T}_n$ be the space of type $\mathcal{T}$ with $m_n = n$ and $\tau_n = \lfloor n^{p_2 - 1} \rfloor$.
3. Let $\mathfrak{S}'_n$ and $\mathcal{T}'_n$ be the rescaled versions of $\mathfrak{S}_n$ and $\mathcal{T}_n$.

Finally, by using the spaces of types $\mathfrak{S}$ and $\mathcal{T}$ simultaneously, we can prove the following theorem.

Theorem 3

Let $1 \leq p_1 < p_2 < \infty$. Then there exists a non-doubling metric measure space for which $\mathcal{M}^c$ and $\mathcal{M}$ are of strong type (p, p) if and only if $p \in [p_1, \infty]$ and $p \in [p_2, \infty]$, respectively.

Proof.

1. For each $n \in \{1, 3, 5, \dots\}$ let $\mathfrak{S}_n$ be the space of type $\mathfrak{S}$ with $m_n = n$ and $\tau_n = \lfloor n^{p_1 - 1} \rfloor$.
2. For each $n \in \{2, 4, 6, \dots\}$ let $\mathcal{T}_n$ be the space of type $\mathcal{T}$ with $m_n = n$ and $\tau_n = \lfloor n^{p_2 - 1} \rfloor$.
3. Let $\mathfrak{S}'_n$ and $\mathcal{T}'_n$ be the rescaled versions of $\mathfrak{S}_n$ and $\mathcal{T}_n$.
4. Apply Theorem SCT to the family $\{\mathfrak{S}'_1, \mathcal{T}'_2, \mathfrak{S}'_3, \mathcal{T}'_4, \dots\}$.

□

Thank you!