Schauder estimates for non-local operators

Franziska Kühn
(Institut de Mathématiques de Toulouse)

Probability and Analysis 2019
May 23, 2019

Outline

Introduction

Aim: Study pointwise regularity of solutions to the equation

$$Af = g$$

where A is an integro-differential operator ($\to$ infinitesimal generator of a Lévy(-type) process).

Introduction

Aim: Study pointwise regularity of solutions to the equation

$$Af = g$$

where A is an integro-differential operator ($\to$ infinitesimal generator of a Lévy(-type) process).

Questions:

- How regular is $f \in \mathcal{D}(A)$?
- If g has a certain regularity then what additional information do we get on the regularity of f?

Introduction

Aim: Study pointwise regularity of solutions to the equation

$$Af = g$$

where A is an integro-differential operator ($\to$ infinitesimal generator of a Lévy(-type) process).

Questions:

- How regular is $f \in \mathcal{D}(A)$?
- If g has a certain regularity then what additional information do we get on the regularity of f?

Formally,

$$f = A^{-1}g$$

i.e. we are interested in the smoothing properties of A^{-1}.

Example I

Laplacian:

$$Af(x) = \frac{1}{2}\Delta f(x)$$

. . . appears as infinitesimal generator of Brownian motion.

Example I

Laplacian:

$$Af(x) = \frac{1}{2}\Delta f(x)$$

. . . appears as infinitesimal generator of Brownian motion. Known:

- If $f \in \mathcal{D}(A)$ then f is "almost twice differentiable":

$$\|f\|_{\mathcal{C}_b^2(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_\infty).$$

Example I

Laplacian:

$$Af(x) = \frac{1}{2}\Delta f(x)$$

. . . appears as infinitesimal generator of Brownian motion. Known:

- If $f \in \mathcal{D}(A)$ then f is "almost twice differentiable":

$$\|f\|_{\mathcal{C}_b^2(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_\infty).$$

$C_b^k(\mathbb{R}^d) \subset \mathcal{C}_b^k(\mathbb{R}^d)$ for $k \in \mathbb{N}$ and $C_b^\alpha(\mathbb{R}^d) = \mathcal{C}_b^\alpha(\mathbb{R}^d)$ for $\alpha \notin \mathbb{N}$

Example I

Laplacian:

$$Af(x) = \frac{1}{2}\Delta f(x)$$

... appears as infinitesimal generator of Brownian motion. Known:

- If $f \in \mathcal{D}(A)$ then f is "almost twice differentiable":

$$\|f\|_{\mathcal{C}_b^2(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_\infty).$$

- If $Af = g \in \mathcal{C}_b^\kappa(\mathbb{R}^d)$ for some $\kappa > 0$ then

$$\|f\|_{\mathcal{C}_b^{2+\kappa}(\mathbb{R}^d)} \leq M_\kappa(\|f\|_\infty + \|Af\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)}).$$

$C_b^k(\mathbb{R}^d) \subset \mathcal{C}_b^k(\mathbb{R}^d)$ for $k \in \mathbb{N}$ and $C_b^\alpha(\mathbb{R}^d) = \mathcal{C}_b^\alpha(\mathbb{R}^d)$ for $\alpha \notin \mathbb{N}$

Example II

Fractional Laplacian:

$$\begin{aligned} Af(x) &= -(-\Delta)^{\alpha/2} f(x) \\ &:= c \int_{y \neq 0} \left(f(x+y) - f(x) - \nabla f(x) y \mathbb{1}_{(0,1)}(|y|) \right) \frac{1}{|y|^{d+\alpha}} \, dy \end{aligned}$$

. . . appears as infinitesimal generator of isotropic α-stable Lévy process.

Example II

Fractional Laplacian:

$$
\begin{aligned}
Af(x) &= -(-\Delta)^{\alpha/2} f(x) \\
&:= c \int_{y \neq 0} \left(f(x+y) - f(x) - \nabla f(x) y \mathbb{1}_{(0,1)}(|y|) \right) \frac{1}{|y|^{d+\alpha}} \, dy
\end{aligned}
$$

. . . appears as infinitesimal generator of isotropic α-stable Lévy process.

Known:

- If $f \in \mathcal{D}(A)$ then

$$
\|f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_\infty).
$$

Example II

Fractional Laplacian:

$$\begin{aligned} Af(x) &= -(-\Delta)^{\alpha/2} f(x) \\ &:= c \int_{y \neq 0} \left(f(x+y) - f(x) - \nabla f(x) y \mathbb{1}_{(0,1)}(|y|) \right) \frac{1}{|y|^{d+\alpha}} \, dy \end{aligned}$$

...appears as infinitesimal generator of isotropic α-stable Lévy process.

Known:

- If $f \in \mathcal{D}(A)$ then

$$\|f\|_{\mathcal{C}_b^{\alpha}(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_\infty).$$

- If $Af = g \in \mathcal{C}_b^{\kappa}(\mathbb{R}^d)$ for some $\kappa > 0$ then

$$\|f\|_{\mathcal{C}_b^{\alpha+\kappa}(\mathbb{R}^d)} \leq M_\kappa(\|f\|_\infty + \|Af\|_{\mathcal{C}_b^{\kappa}(\mathbb{R}^d)}).$$

Infinitesimal generator

...appears in the study of Markov processes. Idea:

$$Af = \frac{d}{dt} P_t f \Big|_{t=0} = \lim_{t \to 0} \frac{P_t f - f}{t}$$

where

$$P_t f(x) := \mathbb{E}^x f(X_t) \overset{\text{Lévy}}{=} \mathbb{E}^0 f(x + X_t).$$

is the semigroup.

Infinitesimal generator

...appears in the study of Markov processes. Idea:

$$Af = \frac{d}{dt} P_t f \Big|_{t=0} = \lim_{t \to 0} \frac{P_t f - f}{t}$$

where

$$P_t f(x) := \mathbb{E}^x f(X_t) \overset{\text{Lévy}}{=} \mathbb{E}^0 f(x + X_t).$$

is the semigroup. Hence,

$$\mathbb{E}^x f(X_t) \approx f(x) + tAf(x) \quad \text{for small } t$$

i.e. A describes small-time asymptotics.

Generator of a Lévy process

Theorem

Let $(X_t)_{t\geq 0}$ be a Lévy process. If $f \in C_c^\infty(\mathbb{R}^d)$ then

$$Af(x) = b \cdot \nabla f(x) + \frac{1}{2}\operatorname{tr}(Q \cdot \nabla^2 f(x)) + \int_{y\neq 0} (f(x+y) - f(x) - \nabla f(x) \cdot y \mathbb{1}_{(0,1)}(|y|))\, \nu(dy)$$

Generator of a Lévy process

Theorem

Let $(X_t)_{t\geq 0}$ *be a Lévy process. If* $f \in C_c^\infty(\mathbb{R}^d)$ *then*

$$Af(x) = b \cdot \nabla f(x) + \frac{1}{2}\operatorname{tr}(Q \cdot \nabla^2 f(x)) + \int_{y\neq 0} (f(x+y) - f(x) - \nabla f(x) \cdot y \mathbb{1}_{(0,1)}(|y|))\,\nu(dy)$$

where the Lévy triplet (b, Q, ν) *consists of*

- $b \in \mathbb{R}^d$ *(drift vector),*
- $Q \in \mathbb{R}^{d\times d}$ *symmetric and positive semidefinite (diffusion matrix)*
- *a measure* ν *with* $\int_{y\neq 0} \min\{1, |y|^2\}\,\nu(dy) < \infty$ *(Lévy measure)*

Generator of a Lévy process

Theorem

Let $(X_t)_{t\geq 0}$ be a Lévy process. If $f \in C_c^\infty(\mathbb{R}^d)$ then

$$Af(x) = b\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}(Q\cdot\nabla^2 f(x)) + \int_{y\neq 0}(f(x+y)-f(x)-\nabla f(x)\cdot y\mathbb{1}_{(0,1)}(|y|))\,\nu(dy)$$

where the Lévy triplet (b, Q, ν) consists of

- $b \in \mathbb{R}^d$ *(drift vector),*
- $Q \in \mathbb{R}^{d\times d}$ *symmetric and positive semidefinite (diffusion matrix)*
- *a measure ν with $\int_{y\neq 0}\min\{1,|y|^2\}\,\nu(dy) < \infty$ (Lévy measure)*

Equivalent characterization via the characteristic exponent

$$\psi(\xi) = -ib\cdot\xi + \frac{1}{2}\xi\cdot Q\xi + \int_{y\neq 0}(1-e^{iy\cdot\xi}+iy\cdot\xi\mathbb{1}_{(0,1)}(|y|))\,\nu(dy)$$

Hölder–Zygmund space

$$\mathcal{C}_b^\alpha(\mathbb{R}^d) := \left\{ f \in C_b(\mathbb{R}^d);\ \|f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} := \|f\|_\infty + \sup_{x \in \mathbb{R}^d} \sup_{0<|h|\leq 1} \frac{|\Delta_h^k f(x)|}{|h|^\alpha} < \infty \right\}$$

where $k = \lfloor \alpha \rfloor + 1$ and

$$\Delta_h := f(x+h) - f(x) \qquad \Delta_h^k f(x) := \Delta_h(\Delta_h^{k-1} f)(x).$$

Hölder–Zygmund space

$$\mathcal{C}_b^\alpha(\mathbb{R}^d) := \left\{ f \in C_b(\mathbb{R}^d);\ \|f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} := \|f\|_\infty + \sup_{x\in\mathbb{R}^d} \sup_{0<|h|\leq 1} \frac{|\Delta_h^k f(x)|}{|h|^\alpha} < \infty \right\}$$

where $k = \lfloor \alpha \rfloor + 1$ and

$$\Delta_h := f(x+h) - f(x) \qquad \Delta_h^k f(x) := \Delta_h(\Delta_h^{k-1} f)(x).$$

Theorem

1. $C_b^k(\mathbb{R}^d) \subset \mathcal{C}_b^k(\mathbb{R}^d)$ *for all* $k \in \mathbb{N}$,
2. $C_b^\alpha(\mathbb{R}^d) = \mathcal{C}_b^\alpha(\mathbb{R}^d)$ *for* $\alpha \in (0,\infty)\backslash\mathbb{N}$.

Schauder estimates for Lévy generators

How to obtain Schauder estimates for solutions to

$$Af = g$$

where A is the generator of a Lévy process?

Schauder estimates for Lévy generators

How to obtain Schauder estimates for solutions to

$$Af = g$$

where A is the generator of a Lévy process?

Known results:

- Bass '09, Ros-Oton & Serra '16: stable operators
- Bae & Kassmann '15: $\nu(dy) = 1/(|y|^d \varphi(y))$
- classical theory for pseudo-differential operators: $\int_{|y|>1} |y|^n \, \nu(dy) < \infty$ for $n \gg 1$

Schauder estimates for Lévy generators

$$\|f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \leq c(\|f\|_\infty + \|Af\|_\infty)$$

Important question: How to measure regularizing effect, i.e. how to find $\alpha > 0$?

Schauder estimates for Lévy generators

$$\|f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \leq c(\|f\|_\infty + \|Af\|_\infty)$$

Important question: How to measure regularizing effect, i.e. how to find $\alpha > 0$?

Idea: Use gradient estimates for the transition density

$$\int_{\mathbb{R}^d} |\nabla p_t(x)| \, dx \leq c t^{-1/\alpha}, \qquad t \in (0,1)$$

Schauder estimates for Lévy generators

$$\|f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \leq c(\|f\|_\infty + \|Af\|_\infty)$$

Important question: How to measure regularizing effect, i.e. how to find $\alpha > 0$?

Idea: Use gradient estimates for the transition density

$$\int_{\mathbb{R}^d} |\nabla p_t(x)|\, dx \leq ct^{-1/\alpha}, \qquad t \in (0,1)$$

Equivalent to

$$\|\nabla P_t u\|_\infty \leq c' t^{-1/\alpha} \|u\|_\infty, \qquad u \in \mathcal{B}_b(\mathbb{R}^d), t \in (0,1).$$

Schauder estimates for Lévy generators

Theorem (K. '19)

Let $(X_t)_{t\geq 0}$ *be a Lévy process with generator* $(A, \mathcal{D}(A))$. *Assume that*

$$\lim_{|\xi|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\log(1+|\xi|)} = \infty$$

and that the transition density p_t *satisfies the gradient estimate*

$$\int_{\mathbb{R}^d} |\nabla p_t(x)|\,dx \leq ct^{-1/\alpha}, \qquad t \in (0,1)$$

for some $\alpha > 0$ *and* $c > 0$.

Schauder estimates for Lévy generators

Theorem (K. '19)

Let $(X_t)_{t\geq 0}$ be a Lévy process with generator $(A, \mathcal{D}(A))$. Assume that

$$\lim_{|\xi|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\log(1+|\xi|)} = \infty$$

and that the transition density p_t satisfies the gradient estimate

$$\int_{\mathbb{R}^d} |\nabla p_t(x)|\, dx \leq ct^{-1/\alpha}, \qquad t \in (0,1)$$

for some $\alpha > 0$ and $c > 0$. Then

- $\|f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_\infty)$ *for any $f \in \mathcal{D}(A)$,*

Schauder estimates for Lévy generators

Theorem (K. '19)

Let $(X_t)_{t\geq 0}$ be a Lévy process with generator $(A, \mathcal{D}(A))$. Assume that

$$\lim_{|\xi|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\log(1+|\xi|)} = \infty$$

and that the transition density p_t satisfies the gradient estimate

$$\int_{\mathbb{R}^d} |\nabla p_t(x)|\, dx \leq ct^{-1/\alpha}, \qquad t \in (0,1)$$

for some $\alpha > 0$ and $c > 0$. Then

- $\|f\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_\infty)$ *for any* $f \in \mathcal{D}(A)$,
- $\|f\|_{\mathcal{C}_b^{\kappa+\alpha}(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)})$ *if* $Af = g \in \mathcal{C}_b^\kappa(\mathbb{R}^d)$ *for* $\kappa > 0$

Idea of proof

$$\int_{\mathbb{R}^d} |\nabla p_t(x)| \, dx \leq c t^{-1/\alpha}, \qquad t \in (0,1)$$

1. Gradient estimate implies

$$\int_{\mathbb{R}^d} |\partial_x^\gamma p_t(x)| \, dx \leq c t^{-|\gamma|/\alpha}, \qquad t \in (0,1), \gamma \in \mathbb{N}_0^d$$

Idea of proof

$$\int_{\mathbb{R}^d} |\nabla p_t(x)| \, dx \leq c t^{-1/\alpha}, \qquad t \in (0,1)$$

1. Gradient estimate implies

$$\int_{\mathbb{R}^d} |\partial_x^\gamma p_t(x)| \, dx \leq c t^{-|\gamma|/\alpha} e^{mt}, \qquad t > 0, \gamma \in \mathbb{N}_0^d$$

Idea of proof

$$\int_{\mathbb{R}^d} |\nabla p_t(x)| \, dx \leq c t^{-1/\alpha}, \qquad t \in (0,1)$$

1. Gradient estimate implies

$$\int_{\mathbb{R}^d} |\partial_x^\gamma p_t(x)| \, dx \leq c t^{-|\gamma|/\alpha} e^{mt}, \qquad t > 0, \gamma \in \mathbb{N}_0^d$$

2. regularizing property of the resolvent $R_\lambda u = \int_{(0,\infty)} e^{-\lambda t} P_t u \, dt$:

Idea of proof

$$\int_{\mathbb{R}^d} |\nabla p_t(x)| \, dx \leq ct^{-1/\alpha}, \qquad t \in (0,1)$$

1. Gradient estimate implies

$$\int_{\mathbb{R}^d} |\partial_x^\gamma p_t(x)| \, dx \leq ct^{-|\gamma|/\alpha} e^{mt}, \qquad t > 0, \gamma \in \mathbb{N}_0^d$$

2. regularizing property of the resolvent $R_\lambda u = \int_{(0,\infty)} e^{-\lambda t} P_t u \, dt$:

$$u \in \mathcal{C}_b^\kappa(\mathbb{R}^d) \implies R_\lambda u \in \mathcal{C}_b^{\kappa+\alpha}(\mathbb{R}^d)$$

for all $\kappa \geq 0$ and $\lambda > m$

Idea of proof

$$\int_{\mathbb{R}^d} |\nabla p_t(x)| \, dx \leq c t^{-1/\alpha}, \qquad t \in (0,1)$$

1. Gradient estimate implies

$$\int_{\mathbb{R}^d} |\partial_x^\gamma p_t(x)| \, dx \leq c t^{-|\gamma|/\alpha} e^{mt}, \qquad t > 0, \gamma \in \mathbb{N}_0^d$$

2. regularizing property of the resolvent $R_\lambda u = \int_{(0,\infty)} e^{-\lambda t} P_t u \, dt$:

$$u \in \mathcal{C}_b^\kappa(\mathbb{R}^d) \implies R_\lambda u \in \mathcal{C}_b^{\kappa+\alpha}(\mathbb{R}^d)$$

for all $\kappa \geq 0$ and $\lambda > m$, and

$$\|R_\lambda u\|_{\mathcal{C}_b^{\alpha+\kappa}(\mathbb{R}^d)} \leq C \|u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)}, \qquad u \in \mathcal{C}_b^\kappa(\mathbb{R}^d), \kappa \geq 0$$

Idea of proof

$$\int_{\mathbb{R}^d} |\nabla p_t(x)| \, dx \leq c t^{-1/\alpha}, \qquad t \in (0,1)$$

1. Gradient estimate implies

$$\int_{\mathbb{R}^d} |\partial_x^\gamma p_t(x)| \, dx \leq c t^{-|\gamma|/\alpha} e^{mt}, \qquad t > 0, \gamma \in \mathbb{N}_0^d$$

2. regularizing property of the resolvent $R_\lambda u = \int_{(0,\infty)} e^{-\lambda t} P_t u \, dt$:

$$u \in \mathcal{C}_b^\kappa(\mathbb{R}^d) \implies R_\lambda u \in \mathcal{C}_b^{\kappa+\alpha}(\mathbb{R}^d)$$

for all $\kappa \geq 0$ and $\lambda > m$, and

$$\|R_\lambda u\|_{\mathcal{C}_b^{\alpha+\kappa}(\mathbb{R}^d)} \leq C \|u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)}, \qquad u \in \mathcal{C}_b^\kappa(\mathbb{R}^d), \kappa \geq 0$$

3. If $f \in \mathcal{D}(A)$ then $f = R_\lambda u$ for $u := \lambda f - Af$.

Example I

Example

Let $(X_t)_{t\geq 0}$ be a Lévy process with triplet (b, Q, ν) and generator $(A, \mathcal{D}(A))$. If Q is positive definite, then

$$\|f\|_{\mathcal{C}_b^{\kappa+2}(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_{\mathcal{C}_b^{\kappa}(\mathbb{R}^d)})$$

whenever $Af = g \in \mathcal{C}_b^{\kappa}(\mathbb{R}^d)$ for some $\kappa \geq 0$. In particular, $\mathcal{D}(A) \subseteq \mathcal{C}_b^2(\mathbb{R}^d)$.

Example II

Example

Let $(X_t)_{t\geq 0}$ be a pure-jump Lévy process with

$$\nu(A) \geq \int_0^{r_0} \int_{\mathbb{S}^{d-1}} \mathbb{1}_A(r\theta) r^{-1-\alpha} \, \mu(d\theta) \, dr + \int_{r_0}^{\infty} \int_{\mathbb{S}^{d-1}} \mathbb{1}_A(r\theta) r^{-1-\beta} \, \mu(d\theta) \, dr, \quad A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$$

for $\alpha \in (0,2)$, $\beta \in (0,\infty]$ and a finite measure μ on the unit sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ which is non-degenerate.

Example II

Example

Let $(X_t)_{t\geq 0}$ be a pure-jump Lévy process with

$$\nu(A) \geq \int_0^{r_0} \int_{\mathbb{S}^{d-1}} \mathbb{1}_A(r\theta) r^{-1-\alpha} \, \mu(d\theta) \, dr + \int_{r_0}^{\infty} \int_{\mathbb{S}^{d-1}} \mathbb{1}_A(r\theta) r^{-1-\beta} \, \mu(d\theta) \, dr, \quad A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$$

for $\alpha \in (0,2)$, $\beta \in (0,\infty]$ and a finite measure μ on the unit sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ which is non-degenerate. Then

$$\|f\|_{\mathcal{C}_b^{\kappa+\alpha}(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_{\mathcal{C}_b^{\kappa}(\mathbb{R}^d)})$$

whenever $Af = g \in \mathcal{C}_b^{\kappa}(\mathbb{R}^d)$ for some $\kappa \geq 0$. In particular, $\mathcal{D}(A) \subseteq \mathcal{C}_b^{\alpha}(\mathbb{R}^d)$.

Example II

Example

Let $(X_t)_{t\geq 0}$ be a pure-jump Lévy process with

$$\begin{aligned}\nu(A) \geq \int_0^{r_0}\int_{\mathbb{S}^{d-1}} \mathbb{1}_A(r\theta) r^{-1-\alpha}\,\mu(d\theta)\,dr \\ + \int_{r_0}^{\infty}\int_{\mathbb{S}^{d-1}} \mathbb{1}_A(r\theta) r^{-1-\beta}\,\mu(d\theta)\,dr, \quad A \in \mathcal{B}(\mathbb{R}^d\backslash\{0\})\end{aligned}$$

for $\alpha \in (0,2)$, $\beta \in (0,\infty]$ and a finite measure μ on the unit sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ which is non-degenerate. Then

$$\|f\|_{\mathcal{C}_b^{\kappa+\alpha}(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_{\mathcal{C}_b^{\kappa}(\mathbb{R}^d)})$$

whenever $Af = g \in \mathcal{C}_b^{\kappa}(\mathbb{R}^d)$ for some $\kappa \geq 0$. In particular, $\mathcal{D}(A) \subseteq \mathcal{C}_b^{\alpha}(\mathbb{R}^d)$.

Key: gradient estimates by Schilling, Sztonyk & Wang ('12)

Example III

Corollary

Let $(X_t)_{t\geq 0}$ be a pure-jump Lévy process whose characteristic exponent ψ satisfies

- *the sector condition,* $|\operatorname{Im}\psi(\xi)| \leq c \operatorname{Re}\psi(\xi)$
- $\operatorname{Re}\psi(\xi) \asymp |\xi|^\alpha$ *for* $|\xi| \gg 1$

Then:

Example III

Corollary

Let $(X_t)_{t\geq 0}$ *be a pure-jump Lévy process whose characteristic exponent* ψ *satisfies*

- *the sector condition,* $|\operatorname{Im}\psi(\xi)| \leq c\operatorname{Re}\psi(\xi)$
- $\operatorname{Re}\psi(\xi) \asymp |\xi|^\alpha$ *for* $|\xi| \gg 1$

Then:

1. $\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}_\infty^{\alpha}(\mathbb{R}^d)$ *where* $\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) := \bigcap_{\beta>\alpha} \mathcal{C}_\infty^{\beta}(\mathbb{R}^d)$

Example III

Corollary

Let $(X_t)_{t\geq 0}$ be a pure-jump Lévy process whose characteristic exponent ψ satisfies

- *the sector condition,* $|\operatorname{Im}\psi(\xi)| \leq c \operatorname{Re}\psi(\xi)$
- $\operatorname{Re}\psi(\xi) \asymp |\xi|^\alpha$ *for* $|\xi| \gg 1$

Then:

1. $\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}_\infty^{\alpha}(\mathbb{R}^d)$ *where* $\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) := \bigcap_{\beta>\alpha} \mathcal{C}_\infty^{\beta}(\mathbb{R}^d)$
2. $\|f\|_{\mathcal{C}_b^{\kappa+\alpha}(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_{\mathcal{C}_b^{\kappa}(\mathbb{R}^d)})$ *if* $Af = g \in \mathcal{C}_b^{\kappa}(\mathbb{R}^d)$ *for* $\kappa \geq 0$

Example III

Corollary

Let $(X_t)_{t\geq 0}$ be a pure-jump Lévy process whose characteristic exponent ψ satisfies

- *the sector condition,* $|\operatorname{Im}\psi(\xi)| \leq c \operatorname{Re}\psi(\xi)$
- $\operatorname{Re}\psi(\xi) \asymp |\xi|^\alpha$ *for* $|\xi| \gg 1$

Then:

1. $\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}_\infty^{\alpha}(\mathbb{R}^d)$ *where* $\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) := \bigcap_{\beta>\alpha} \mathcal{C}_\infty^{\beta}(\mathbb{R}^d)$
2. $\|f\|_{\mathcal{C}_b^{\kappa+\alpha}(\mathbb{R}^d)} \leq M(\|f\|_\infty + \|Af\|_{\mathcal{C}_b^{\kappa}(\mathbb{R}^d)})$ *if* $Af = g \in \mathcal{C}_b^{\kappa}(\mathbb{R}^d)$ *for* $\kappa \geq 0$
3. $\mathcal{D}(A)$ *is an algebra, i.e.* $f, g \in \mathcal{D}(A)$ *implies* $f \cdot g \in \mathcal{D}(A)$, *and*

$$A(f \cdot g) = fAg + gAf + \Gamma(f,g)$$

with $\Gamma(f,g)(x) := \int_{y\neq 0} (f(x+y) - f(x))(g(x+y) - g(x))\, \nu(dy)$.

Remarks

The presented Schauder estimates

- ...are sharp.
- ...hold more generally for functions f in the Favard space of order 1,

$$F_1 := \left\{ f \in \mathcal{B}_b(\mathbb{R}^d);\ \sup_{t \in (0,1)} \sup_{x \in \mathbb{R}^d} \frac{|\mathbb{E} f(x+X_t) - f(x)|}{t} < \infty \right\}$$

Schauder estimates for Lévy-type operators

Q: Can the results be extended to Lévy-type operators

$$Af(x) = b(x) \cdot \nabla f(x) + \frac{1}{2} \operatorname{tr}(Q(x) \cdot \nabla^2 f(x)) + \int_{y \neq 0} (f(x+y) - f(x) - \nabla f(x) \cdot y \mathbb{1}_{(0,1)}(|y|)) \, \nu(x, dy)$$

...?

Schauder estimates for operators of variable order

Theorem (K. '19)

Consider

$$Af(x) = c(x) \int_{y \neq 0} (f(x+y) - f(x) - y \nabla f(x) \mathbb{1}_{(0,1)}(|y|)) \frac{1}{|y|^{d+\alpha(x)}} \, dy$$

for $\alpha : \mathbb{R}^d \to (0,2)$ *Hölder continuous with* $\inf_x \alpha(x) > 0$.

Schauder estimates for operators of variable order

Theorem (K. '19)

Consider

$$Af(x) = c(x) \int_{y \neq 0} (f(x+y) - f(x) - y\nabla f(x) \mathbb{1}_{(0,1)}(|y|)) \frac{1}{|y|^{d+\alpha(x)}} \, dy$$

for $\alpha : \mathbb{R}^d \to (0,2)$ *Hölder continuous with* $\inf_x \alpha(x) > 0$. *Then the Schauder estimate*

$$\|f\|_{\mathcal{C}_b^{\alpha(\cdot)-\varepsilon}(\mathbb{R}^d)} \leq M_\varepsilon(\|f\|_\infty + \|Af\|_\infty), \qquad f \in \mathcal{D}(A)$$

holds for $\varepsilon > 0$*; here* $\mathcal{C}_b^{\alpha(\cdot)-\varepsilon}(\mathbb{R}^d)$ *is a Hölder–Zygmund space of variable order.*

Schauder estimates for operators of variable order

Theorem (K. '19)

Consider

$$Af(x) = c(x) \int_{y \neq 0} (f(x+y) - f(x) - y \nabla f(x) \mathbb{1}_{(0,1)}(|y|)) \frac{1}{|y|^{d+\alpha(x)}} \, dy$$

for $\alpha : \mathbb{R}^d \to (0,2)$ *Hölder continuous with* $\inf_x \alpha(x) > 0$. *Then the Schauder estimate*

$$\|f\|_{\mathcal{C}_b^{\alpha(\cdot)-\varepsilon}(\mathbb{R}^d)} \leq M_\varepsilon (\|f\|_\infty + \|Af\|_\infty), \qquad f \in \mathcal{D}(A)$$

holds for $\varepsilon > 0$; *here* $\mathcal{C}_b^{\alpha(\cdot)-\varepsilon}(\mathbb{R}^d)$ *is a Hölder–Zygmund space of variable order. In particular,*

$$\mathcal{D}(A) \subseteq \mathcal{C}_b^{\alpha(\cdot)-\varepsilon}(\mathbb{R}^d), \qquad \varepsilon > 0.$$

References I

- Bae, J., Kassmann, M.: Schauder estimates in generalized Hölder spaces. arXiv 1505.05498.
- Bass, R. F.: Regularity results for stable-like operators. *J. Funct. Anal.* **257** (2009), 2693–2722.
- Knopova, V., Schilling, R.L.: A note on the existence of transition probability densities of Lévy processes. *Forum. Math.* **25** (2013), 125–149.
- Kühn, F.: Schauder Estimates for Equations Associated with Lévy Generators. *Int. Eq. Op. Theory* **91**:10 (2019).
- Kühn, F.: Schauder estimates for Poisson equations associated with non-local Feller generators. arXiv 1902.01760.

References II

- Kühn, F., Schilling, R.L.: On the domain of fractional Laplacians and related generators of Feller processes. *J. Funct. Anal.* **276** (2019), 2397–2439.
- Ros-Oton, X., Serra, J.: Regularity theory for general stable operators. *J. Diff. Equations* **260** (2016), 8675–8715.
- Schilling, R. L., Sztonyk, P., Wang, J.: Coupling property and gradient estimates of Lévy processes via the symbol. *Bernoulli* **18** (2012), 1128–1149.
- Stein, E. M.: *Singular integrals and differentiability properties of functions*. Princeton Univ. Press 1970.
- Triebel, H.: *Interpolation theory, function spaces, differential operators*. North-Holland 1978.