Strong Feller property for SDEs driven by multiplicative cylindrical stable noice

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$Z_t = \left(Z_t^{(1)}, \dots, Z_t^{(d)}\right)$ - cylindrical α-stable process in $\mathbb{R}^d$, $Z_t^{(1)}, \dots, Z_t^{(d)}$ independent one-dimensional symmetric α-stable processes, $\alpha \in (0,2)$.

$$-\sum_{i=1}^{d} \left(-\frac{\partial^2}{\partial x_i^2}\right)^{\alpha/2} f(x) = \sum_{i=1}^{d} \lim_{\zeta \to 0^+} \mathcal{A}_\alpha \int_{|w_i| > \zeta} \frac{[f(x + w_i e_i) - f(x)] \, dw_i}{|w_i|^{1+\alpha}}.$$

$$J_Z(x, B) = \mathcal{A}_\alpha \sum_{i=1}^{d} \int_B \frac{dw_i}{|w_i - x_i|^{1+\alpha}} \prod_{k \neq i} \delta_{x_k}(dw_k).$$

$Y_t = \left(Y_t^{(1)}, \dots, Y_t^{(d)}\right)$ - isotropic α-stable process in $\mathbb{R}^d$.

$$-(-\Delta)^{\alpha/2} f(x) = \lim_{\zeta \to 0^+} \mathcal{A}_{\alpha,d} \int_{|v| > \zeta} \frac{[f(x+v) - f(x)] \, dv}{|v|^{d+\alpha}}.$$

$$J_Y(x, B) = \mathcal{A}_{\alpha,d} \int_B \frac{dv}{|v - x|^{1+\alpha}}.$$

$$Z_t = \begin{pmatrix} Z_t^{(1)} \\ \vdots \\ Z_t^{(d)} \end{pmatrix} \text{ - cylindrical } \alpha\text{-stable process in } \mathbb{R}^d.$$

We assume that $\alpha \in (0,1)$, $d \in \mathbb{N}$, $d \geqslant 2$.

$$dX_t = A(X_{t-})\, dZ_t, \quad X_0 = x. \tag{1}$$

$A(x) = (a_{ij}(x))$, there are constants $\eta_1, \eta_2, \eta_3 > 0$ such that for any $x, y \in \mathbb{R}^d$, $i, j \in \{1, \ldots, d\}$

$$a_{ij}(x) \leqslant \eta_1, \tag{2}$$

$$\det(A(x)) \geqslant \eta_2, \tag{3}$$

$$|a_{ij}(x) - a_{ij}(y)| \leqslant \eta_3 |x - y|. \tag{4}$$

$$dX_t = A(X_{t-})\, dZ_t, \quad X_0 = x, \tag{1}$$

It is well known that SDE (1) has a unique strong solution X_t. The process X_t is a Feller process (Schilling, Schnurr (2010)).
The generator of X is given by

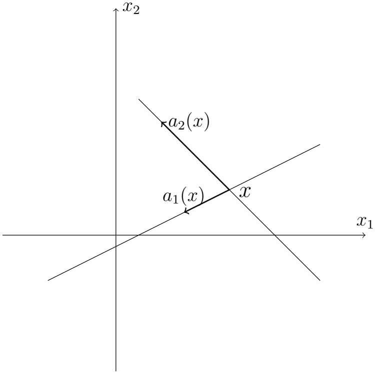
$$\mathcal{K}f(x) = \sum_{i=1}^{d} \lim_{\zeta \to 0^+} \mathcal{A}_\alpha \int_{|w_i|>\zeta} \frac{[f(x + a_i(x)w_i) - f(x)]\; dw_i}{|w_i|^{1+\alpha}}.$$

where $a_i(x) = (a_{1i}(x), \ldots, a_{di}(x))$.

For $d = 2$ the generator of X is given by

$$\mathcal{K}f(x) = \sum_{i=1}^{2} \lim_{\zeta \to 0^+} \mathcal{A}_\alpha \int_{|w_i|>\zeta} \frac{[f(x + a_i(x)w_i) - f(x)]\ dw_i}{|w_i|^{1+\alpha}}.$$

where $a_i(x) = (a_{1i}(x), a_{2i}(x))$.



Main results

$$P_t f(x) = \mathbb{E}^x f(X_t),\ t \geqslant 0,\ x \in \mathbb{R}^d,\ f \in \mathcal{B}_b(\mathbb{R}^d).$$

Theorem (T.K., M. Ryznar, P. Sztonyk)

For any $\gamma \in (0, \alpha)$, $\tau > 0$, $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$ we have

$$|P_t f(x) - P_t f(y)| \leqslant c t^{-\gamma/\alpha} |x - y|^{\gamma} \, ||f||_{\infty}, \tag{5}$$

where c depends on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3, \gamma$.

$P_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y)\, dy$, $t \geqslant 0$, $x \in \mathbb{R}^d$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, (A. Debussche, N. Fournier (2013)).

Remark (T.K., M. Ryznar, P. Sztonyk)

For any $d \geqslant 2$, $\alpha \in (0, 1)$ there exist $\{A(x)\}_{x \in \mathbb{R}^d}$, points $x_0, y_0 \in \mathbb{R}^d$, time $t_0 > 0$ and constants $c, \delta > 0$ such that

$$p(t_0, x_0, y) \geqslant c |y - y_0|^{\alpha - d + 1} \qquad \text{for almost all } \ y \in B(y_0, \delta).$$

Levi's method (parametrix method)

E. Levi (1907)
O. Ladyzenskaja, V. Solonnikov, N. Uralceva (1968)
A. Friedman (1975)
For nonlocal operators:
A. Kochubei (1988)
K. Bogdan, T. Jakubowski (2007)
Z.-Q. Chen, P. Kim, T. Kumagai (2011)
L. Xie, X. Zhang (2014)
Z.-Q. Chen, X. Zhang (2016)
P. Kim, R. Song, Z. Vondraček (2016)
F. Kühn (2017)
T. Grzywny, K. Szczypkowski (2018)
V. Knopova, A. Kulik (2018),

- $$\left(\frac{\partial}{\partial t} - \mathcal{K}\right) p(t,\cdot,y)(x) = 0. \qquad \lim_{t\to 0^+} p(t,x,y)\,dy = \delta_x.$$

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- $$p(t,x,y) = \tilde{p}(t,x,y) + r(t,x,y). \qquad \lim_{t\to 0^+} \tilde{p}(t,x,y)\,dy = \delta_x.$$

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- $$\left(\frac{\partial}{\partial t} - \mathcal{K}\right) r(t,\cdot,y)(x) = \underbrace{-\left(\frac{\partial}{\partial t} - \mathcal{K}\right) \tilde{p}(t,\cdot,y)(x)}_{q_0(t,x,y)}.$$

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- $$r(t,x,y) = \int_0^t \int_{\mathbb{R}^d} p(t-s,x,z) q_0(s,z,y)\,dz\,ds.$$

- $$\left(\frac{\partial}{\partial t} - \mathcal{K}\right) p(t,\cdot,y)(x) = 0. \qquad \lim_{t\to 0^+} p(t,x,y)\,dy = \delta_x.$$

- $$p(t,x,y) = \tilde{p}(t,x,y) + r(t,x,y). \qquad \lim_{t\to 0^+} \tilde{p}(t,x,y)\,dy = \delta_x.$$

- $$\left(\frac{\partial}{\partial t} - \mathcal{K}\right) r(t,\cdot,y)(x) = \underbrace{-\left(\frac{\partial}{\partial t} - \mathcal{K}\right)\tilde{p}(t,\cdot,y)(x)}_{q_0(t,x,y)}.$$

- $$r(t,x,y) = \int_0^t \int_{\mathbb{R}^d} p(t-s,x,z)q_0(s,z,y)\,dz\,ds.$$

- $$p(t,x,y) = \tilde{p}(t,x,y) + \int_0^t \int_{\mathbb{R}^d} p(t-s,x,z)q_0(s,z,y)\,dz\,ds.$$

$$q_{n+1}(t,x,y) = \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z) q_n(s,z,y)\, dz\, ds.$$

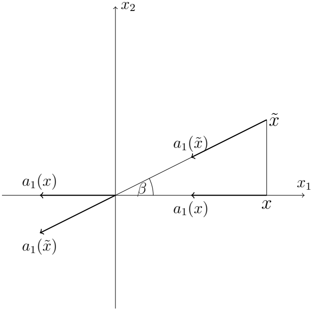
$$p(t,x,y) = \tilde{p}(t,x,y) + \sum_{n=0}^{\infty} \int_0^t \int_{\mathbb{R}^d} \tilde{p}(t-s,x,z) q_n(s,z,y)\, dz\, ds.$$

$$\mathcal{K} f(x) = \sum_{i=1}^{d} \lim_{\zeta \to 0^+} \mathcal{A}_\alpha \int_{|w_i| > \zeta} \frac{[f(x + a_i(x) w_i) - f(x)]\ dw_i}{|w_i|^{1+\alpha}}.$$

"freezing operator":

$$\mathcal{K}^y f(x) = \sum_{i=1}^{d} \lim_{\zeta \to 0^+} \mathcal{A}_\alpha \int_{|w_i| > \zeta} \frac{[f(x + a_i(y) w_i) - f(x)]\ dw_i}{|w_i|^{1+\alpha}}.$$

$p_y(t,x) \sim \mathcal{K}^y$, $\left(\frac{\partial}{\partial t} - \mathcal{K}^y\right) p_y(t,\cdot)(x) = 0$.
$\tilde{p}(t,x,y) = p_y(t, y-x)$? problem: $\int_{\mathbb{R}^d} p_y(t, y-x)\, dy = \infty$ in some cases.



$d = 2$, $\varepsilon = |x|$ small, $\beta \in (0, \pi/4)$.

$$\beta \approx \sin\beta \approx \frac{|a_1(x) - a_1(\tilde{x})|}{|a_1(x)|} \leqslant c|x - \tilde{x}|.$$

$$\beta \approx \operatorname{tg}\beta = \frac{|x - \tilde{x}|}{\varepsilon}.$$

$$\frac{|x - \tilde{x}|}{\varepsilon} \leqslant c|x - \tilde{x}|.$$

$\delta \in (0,1]$, $\mu : \mathbb{R} \setminus \{0\} \to [0,\infty)$.

$$\mu(x) = \begin{cases} = \mathcal{A}_\alpha |x|^{-1-\alpha}, & \text{for } x \in (0,\delta], \\ \in (0, \mathcal{A}_\alpha |x|^{-1-\alpha}), & \text{for } x \in (\delta, 2\delta], \\ = 0, & \text{for } x \in [2\delta, \infty). \end{cases}$$

μ nonincreasing, convex and C^1 on $(0,\infty)$. $\mu(-x) = \mu(x)$ for $x \in (0,\infty)$.

$\mathcal{G}f(x) = \lim_{\zeta \to 0^+} \int_{|w|>\zeta} (f(x+w) - f(x))\mu(w)\, dw$.

$\frac{\partial}{\partial t} g_t(x) = \mathcal{G} g_t(x), \quad t > 0,\ x \in \mathbb{R}$.

$$h_t(x) = \begin{cases} \frac{t}{(|x| + t^{1/\alpha})^{1+\alpha}} & \text{for } |x| < \varepsilon, \\ c_\varepsilon t^{1+(d-1)/\alpha} e^{-|x|} & \text{for } |x| \geqslant \varepsilon, \end{cases}$$

Lemma

$$g_t(x) \leqslant c h_t(|x|),$$

$$|g_t(x) - g_t(y)| \leqslant c|x-y| \left(\frac{h_t(|x|)}{t^{1/\alpha} + |x|} + \frac{h_t(|y|)}{t^{1/\alpha} + |y|} \right).$$

$$\mathcal{L}f(x) = \sum_{i=1}^{d} \lim_{\zeta\to 0^+} \int_{|w|>\zeta} [f(x+a_i(x)w) - f(x)]\,\mu(w)\,dw,$$

$$\mathcal{L}^y f(x) = \sum_{i=1}^{d} \lim_{\zeta\to 0^+} \int_{|w|>\zeta} [f(x+a_i(y)w) - f(x)]\,\mu(w)\,dw.$$

$p_y(t,x) \sim \mathcal{L}^y$, $\left(\frac{\partial}{\partial t} - \mathcal{L}^y\right) p_y(t,\cdot)(x) = 0.$
$B(x) = A^{-1}(x)$, $B(x) = (b_{ij}(x))$.
$\tilde{p}(t,x,y) = p_y(t,y-x) = \det(B(y)) \prod_{i=1}^{d} g_t(b_i(y)(y-x))$,
$b_i(y) = (b_{i1}(y), \ldots, b_{id}(y))$ - ith row of $B(y)$.

$$\begin{aligned} q_0(t,x,y) &= -\left(\frac{\partial}{\partial t} - \mathcal{L}^x\right) \tilde{p}(t,\cdot,y)(x) \\ &= -(\mathcal{L}^y - \mathcal{L}^x)\, p_y(t,\cdot)(x-y) \\ &= \sum_{i=1}^{d} \int_{\mathbb{R}} [p_y(t,y-x+a_i(x)w) - p_y(t,y-x+a_i(y)w)]\,\mu(w)dw. \end{aligned}$$

For any $t \in (0, \tau]$ we have

$$\int_{\mathbb{R}^d} |q_0(t,x,y)|\,dy \leqslant ct^{-1/2}, \qquad \int_{\mathbb{R}^d} |q_0(t,x,y)|\,dx \leqslant ct^{-1/2},$$

$$|q_0(t,x,y)| \leqslant ct^{-1-(d-1)/\alpha}.$$

$$q_{n+1}(t,x,y) = \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z) q_n(s,z,y)\,dz\,ds.$$

$$u(t,x,y) = \tilde{p}(t,x,y) + \sum_{n=0}^{\infty} \int_0^t \int_{\mathbb{R}^d} \tilde{p}(t-s,x,z) q_n(s,z,y)\,dz\,ds.$$

For any $t \in (0, \tau]$ we have

$$\int_{\mathbb{R}^d} |u(t,x,y)|\,dy \leqslant c, \qquad \int_{\mathbb{R}^d} |u(t,x,y)|\,dx \leqslant c,$$

$$|u(t,x,y)| \leqslant ct^{-d/\alpha}.$$

$U_t f(x) = \int_{\mathbb{R}^d} u(t,x,y) f(y)\,dy$, $f \in \mathcal{B}_b(\mathbb{R}^d)$.
For any $f \in C_0(\mathbb{R}^d)$ in the approximation setting (Knopova, Kulik (2018)) $\left(\frac{\partial}{\partial t} - \mathcal{L}\right) U_t f = 0$.

Long jumps

$\nu(x) = \frac{\mathcal{A}_\alpha}{|x|^{1+\alpha}} - \mu(x), \quad x \in \mathbb{R} \setminus \{0\},$
$\lambda = d \int_{\mathbb{R}} \nu(x)\,dx < \infty.$
$\mathcal{R}f(x) = \sum_{i=1}^{d} \int_{\mathbb{R}} [f(x + a_i(x)w) - f(x)]\, \nu(w)\,dw.$
$\mathcal{N}f(x) = \sum_{i=1}^{d} \int_{\mathbb{R}} [f(x + a_i(x)w)]\, \nu(w)\,dw.$
For any $t \geqslant 0$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $n \geqslant 1$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we define

$$\begin{aligned} \Psi_{0,t}f(x) &= U_t f(x), \\ \Psi_{n,t}f(x) &= \int_0^t U_{t-s}(\mathcal{N}(\Psi_{n-1,s}f))(x)\,ds, \quad n \geqslant 1. \end{aligned}$$

$T_t f(x) = e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t} f(x)$
$T_t \sim \mathcal{K}$. For any $f \in C_0(\mathbb{R}^d)$ in the approximation setting
$\left(\frac{\partial}{\partial t} - \mathcal{K}\right) T_t f = 0$

Martingale problem

We conclude that there is a Feller process $\tilde{X}_t$ with the semigroup T_t and the generator $\mathcal{K}$. Let $\tilde{\mathbb{P}}^x$ be the distribution for the process starting from $x \in \mathbb{R}^d$. For any function $f \in C_b^2(\mathbb{R}^d)$, the process

$$f(\tilde{X}_t) - f(\tilde{X}_0) - \int_0^t \mathcal{K}f(\tilde{X}_s)ds$$

is a $(\tilde{\mathbb{P}}^x, \tilde{\mathcal{F}}_t)$ martingale, where $\tilde{\mathcal{F}}_t$ is a natural filtration. That is, $\tilde{\mathbb{P}}^x$ solves the martingale problem for $(\mathcal{K}, C_b^2(\mathbb{R}^d))$. This gives $P_t = T_t$.
We have

$$T_t f(x) = e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t} f(x),$$

$$\Psi_{0,t} = U_t, \quad \Psi_{n,t} f(x) = \int_0^t U_{t-s}(\mathcal{N}(\Psi_{n-1,s} f))(x)\, ds, \quad n \geqslant 1.$$

$p(t,x,y)$ is unbounded (for some choices of $\{A(x)\}_{x\in\mathbb{R}^d}$)

$d = 2$, $\alpha \in (0,1)$.
$P_t f(x) = T_t f(x) = e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t} f(x)$.

$$\Psi_{1,t} f(x) = \int_0^t U_{t-s}(\mathcal{N}(U_s f))(x)\, ds =$$

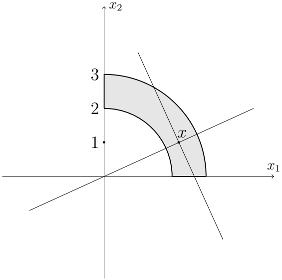
$$\sum_{i=1}^{2} \int_0^t \int_{\mathbb{R}^2} u(t-s,x,z) \int_{\mathbb{R}} \int_{\mathbb{R}^2} u(s, z + a_i(z)w, y) f(y) dy \nu(w) \times dwdzds.$$

For any fixed $t > 0$, $x \in \mathbb{R}^2$ and almost all $y \in \mathbb{R}^d$

$$p(t,x,y) \geqslant e^{-\lambda t} \int_0^t \int_{\mathbb{R}^2} u(t-s,x,z) \int_{\mathbb{R}} u(s, z + a_1(z)w, y) \nu(w) dwdzds.$$

$p(t,x,y)$ is unbounded (for some choices of $\{A(x)\}_{x\in\mathbb{R}^d}$)

Let $d = 2$ and
$D = \{(x_1, x_2) \in \mathbb{R}^2 : 2 \leqslant \sqrt{x_1^2 + x_2^2} \leqslant 3,\ x_1 \geqslant 0, x_2 \geqslant 0\}$.

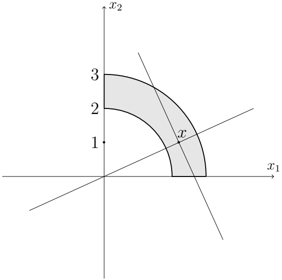


- One can choose $A(x) = A(x_1, x_2)$ so that for any $x \in D$

$$A(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix} = |x|^{-1} \begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix}.$$

$p(t,x,y)$ is unbounded (for some choices of $\{A(x)\}_{x\in\mathbb{R}^d}$)

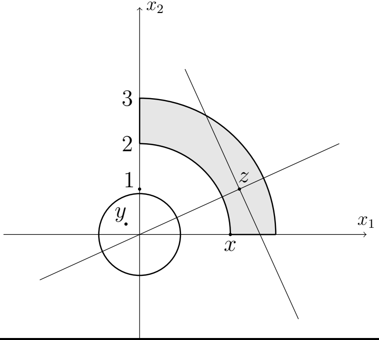
Let $d = 2$ and
$D = \{(x_1, x_2) \in \mathbb{R}^2 : \, 2 \leqslant \sqrt{x_1^2 + x_2^2} \leqslant 3, \; x_1 \geqslant 0, x_2 \geqslant 0\}$.



- One can choose $A(x) = A(x_1, x_2)$ so that for any $x \in D$
  $$A(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix} = |x|^{-1} \begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix}.$$
- We have $a_1(x) = (a_{11}(x), a_{21}(x)) = |x|^{-1}(x_1, x_2) = x/|x|$,
  $a_2(x) = (a_{21}(x), a_{22}(x)) = |x|^{-1}(-x_2, x_1)$.

Let $\alpha \in (0,1)$, $d = 2$, $x = (2,0)$. For any fixed $t > 0$ and almost all $y \in \mathbb{R}^d$

$$p(t,x,y) \geqslant$$
$$e^{-\lambda t} \int_0^t \int_{\mathbb{R}^2} u(t-s,x,z) \int_{\mathbb{R}} u(s, z + a_1(z)w, y)\nu(w)dwdzds.$$



For $z \in D$ and some s, w, y we have $u(s, z + a_1(z)w, y) \geqslant cs^{-2/\alpha}$.

One can show that there exists $t > 0$, $\delta > 0$ such that for almost all $y \in B(0,\delta)$ we have $p(t,x,y) \geqslant c|y|^{\alpha-1}$.

Some related papers

R. Bass, Z.-Q. Chen (2006), (2010)

J. Wang, X. Zhang (2013)

A. Debussche, N. Fournier (2013)

J. Chaker (2016), (2018)

K. Bogdan, V. Knopova, P. Sztonyk (2017)

V. Knopova, A. Kulik (2018)

J. Chaker, M. Kassmann (2018)

M. Liang, J. Wang (2018)