On the nonlocal Dirichlet problem

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Based on joint work with Tomasz Grzywny and Moritz Kassmann.

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HR EXCELLENCE IN RESEARCH

Classical case - overview

- Weyl's lemma: if $f \in C^\infty(D)$, $u \in \mathcal{D}'(D)$ is a Schwartz distribution satisfying $\Delta u = f$ in distributional sense, then $u \in C^\infty(D)$ and $\Delta u = f$ in D.

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- This can be extended for second order elliptic equations, for which coefficients of the differential operator and non-homogeneous term satisfy Dini criterion,
- On the other hand, for $d > 1$ the continuity of f is not enough, i.e. one can construct a continuous function $f \colon B_1 \mapsto \mathbb{R}$ such that $\Delta u = f$ in the distributional sense, but $u \notin C^2_{\mathrm{loc}}(B_1)$.

Problem setting

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Let $\nu : \mathbb{R}^d \setminus \{0\} \mapsto [0, \infty)$ be a function satisfying

$$\int (1 \wedge |h|^2)\nu(h)\,\mathrm{d}h < \infty.$$

The function ν induces a measure $\nu(\mathrm{d}h) = \nu(h)\,\mathrm{d}h$, which is called *the Lévy measure*.

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$$\mathscr{L}u(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \big(u(x+y) - u(x)\big)\nu(y)\,\mathrm{d}y.$$

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- study distributional solutions to nonlocal boundary value problem of the form

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$$\begin{aligned} \mathscr{L}u &= f \quad \text{in } D, \\ u &= g \quad \text{in } D^c, \end{aligned}$$

- provide sufficient conditions for the twice differentiability in the classical sense.

Results for $-(-\Delta)^{\alpha/2}$

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For $\alpha \in (0,2)$ and $\nu(\mathrm{d}h) = c_{d,\alpha}|h|^{-d-\alpha}\,\mathrm{d}h$, the operator $\mathscr{L}$ equals $-(-\Delta)^{\alpha/2}$ on $C_b^2(\mathbb{R}^d)$.

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⋮

And many more!

Main assumption

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$h \mapsto \nu(h)$ is a non-increasing radial function and there exists a Lévy measure ν^* with density $\nu^*(x)$ s.t. $\nu(r) \leqslant \nu^*(r)$ for $r \geqslant r_0$ and

$$\nu^*(r) \leqslant C\nu^*(r+1), \quad r \geqslant r_0, \tag{1}$$

for some $r_0, C \geqslant 1$.

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Given an open set $D \subseteq \mathbb{R}^d$, denote by $\mathcal{L}^1(D)$ the vector space of all Borel functions $u \in L^1_{\mathrm{loc}}$ satisfying

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The fractional Laplace operator $-(-\Delta)^{\alpha/2}$

$$\mathcal{L}^1(\mathbb{R}^d) = \left\{ u \text{ Borel} : \int_{\mathbb{R}^d} \frac{\left|u(x)\right|}{\left(1+|x|\right)^{d+\alpha}} \,\mathrm{d}x < \infty \right\}.$$

Comments on the space $\mathcal{L}(\mathbb{R}^d)$

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$$\int_{|x|>5R} |u(x)| \nu^*(|x|) \, \mathrm{d}x < \infty \quad \not\Rightarrow \quad \int_{|x|>5R} |u(x)| \nu^*(|x| - 1) \, \mathrm{d}x < \infty.$$

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Isotropic unimodal Lévy process

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$\Rightarrow$ isotropic unimodal Lévy process X_t with transition density

$$p_t(x,y) = p_t(y-x), \text{ i.e. } \mathbb{P}^x(X_t \in A) = \int_A p_t(x,y)\,\mathrm{d}y.$$

The Poisson operator

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$$P_D[g](x) = \int_{D^c} g(z) P_D(x, \mathrm{d}z), \qquad x \in D,$$

if the integral exists. For $x \in D^c$ we set $P_D[g](x) = g(x)$.

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The mean-value property

We say that a function $g\colon \mathbb{R}^d \mapsto \mathbb{R}$ satisfies *the mean-value property* in an open set $D \subseteq \mathbb{R}$, if

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If g has the mean-value property in every open bounded set whose closure is contained in D, then we say that g has the mean value property inside D.

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Let $g \in \mathcal{L}^1(\mathbb{R}^d)$, D be an open bounded set. Suppose (1) holds and

(A) ν is twice continuously differentiable and there is a positive constant C such that

$$|\nu'(r)|, |\nu''(r)| \leqslant C\nu^*(r), \quad r \geqslant r_0.$$

If g has the mean-value property inside D, then $g \in C^2_{\mathrm{loc}}(D)$.

Equivalent definitions of harmonicity

- „probabilistic" definition: u is harmonic with respect to X_t if has the mean-value property inside D, i.e. $P_D[u] = u$.

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But $\mathscr{L}$ and $X = (X_t)_{t \geqslant 0}$ are closely related, so both definitions may be *expected* to be equivalent...

This is the case when $\mathscr{L} = -(-\Delta)^{\alpha/2}$ (resp. $\nu(x) = c_{d,\alpha}|x|^{-d-\alpha}$), see K. Bogdan, T. Byczkowski, 1999.

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Theorem (T. Grzywny, ŁL, M. Kassmann)

Let D be an open set and $u \in \mathcal{L}^1(\mathbb{R}^d)$. Then u has the mean-value property inside D if and only if $\mathscr{L}u = 0$ in D in distributional sense.

Green function and Green operator

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One can consider a process killed outside an open set D. Let $p_t^D(x,y)$ be its transition density.

We define *a Green function* for the set D

$$G_D(x,y) = \int_0^\infty p_t^D(x,y)\mathrm{d}t,$$

and *the Green operator*

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Kato class $\mathcal{K}$

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We say that a Borel function f belongs to the Kato class $\mathcal{K}$ if it satisfies the following condition

$$\lim_{r\to 0^+}\left[\sup_{x\in\mathbb{R}^d}\int_0^r P_t|f|(x)\mathrm{d}t\right]=0.$$

We say that $f\in\mathcal{K}(D)$, where D is an open set, if $f\mathbf{1}_D\in\mathcal{K}$.

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Let V, D be open sets, $\overline{V}\subseteq D$ and $\rho := \mathrm{dist}(\mathrm{V},\partial\mathrm{D})$. Suppose $f\in\mathcal{K}(D\setminus\overline{V})$. Then $G_D[f]$ is bounded in the set $V_1 := \{x\in D\setminus V : \delta_D(x)<\rho/2\}$.

Theorem (T. Grzywny, ŁL, M. Kassmann)

Let D be a bounded open set. Suppose $f \in L^1(D)$ and $g \in \mathcal{L}^1(D^c)$. Let $u \in \mathcal{L}^1(\mathbb{R}^d)$ be a distributional solution of the Dirichlet problem

$$\text{(P)} \qquad \begin{aligned} \mathscr{L} u &= f \quad \text{in } D, \\ u &= g \quad \text{in } D^c. \end{aligned}$$

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Then $u(x) + G_D[f](x)$ satisfies the mean-value property inside D. Furthermore, if D is a Lipschitz domain and there exists $V \subset\subset D$ such that f and $g * \nu$ belong to the Kato class $\mathcal{K}(D \setminus \overline{V})$, then there is a unique solution which is bounded close to the boundary of D

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Potential kernels

If $\int_{B_1} \frac{\mathrm{d}\xi}{\psi(\xi)} < \infty$, we can define a *potential kernel*

$$U(x,y) = \int_0^\infty p_t(x,y)\,\mathrm{d}t.$$

For $d \geqslant 3$ the potential kernel always exists.

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For $d \geqslant 3$ the potential kernel always exists. If this is not the case, one can consider a *compensated potential kernel*

$$W_{x_0}(x,y) = \int_0^\infty \big(p_t(x-y) - p_t(x_0)\big)\,\mathrm{d}t, \tag{2}$$

for some fixed $x_0 \in \mathbb{R}^d$. If $d = 1$ and $\int_0^\infty \frac{1}{1+\psi(\xi)}\mathrm{d}\xi < \infty$, then we can set $x_0 = 0$.

Potential kernels

Let W_1 be (2) for $x_0 = (0, \dots, 0, 1) \in \mathbb{R}^d$. We set

$$G(x) = \begin{cases} U(x), & \text{if } \int_{B_1} \frac{\mathrm{d}\xi}{\psi(\xi)} < \infty, \\ W_0(x), & \text{if } d = 1, \int_0^1 \frac{\mathrm{d}\xi}{\psi(\xi)} = \infty \text{ and } \int_0^\infty \frac{\mathrm{d}\xi}{1+\psi(\xi)} < \infty, \\ W_1(x), & \text{otherwise.} \end{cases}$$

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The (fractional) Laplace operator

In case of $\mathscr{L} = \Delta$ we have

$$G(x) = \begin{cases} c_d |x|^{2-d}, & d \geqslant 3, \\ \frac{1}{\pi} \ln \frac{1}{|x|}, & d = 2, \\ |x|, & d = 1. \end{cases}$$

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For $\mathscr{L} = -(-\Delta)^{\alpha/2}$ we have $G(x) = c_{d,\alpha} |x|^{\alpha - d}$, $d \neq \alpha$.

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Assumption 2

(G) $G \in C^2(\mathbb{R}^d \setminus \{0\})$ and there exists a non-increasing function $S\colon (0,\infty) \mapsto [0,\infty)$ and $r_0 > 0$ such that

(i) if $\int_0^1 |G'(t)| t^{d-1} \mathrm{d}t = \infty$, then

$$G(r), |G'(r)|, r|G''(r)| \leqslant S(r), \quad r \geqslant r_0.$$

(ii) if $\int_0^1 |G'(t)| t^{d-1} \mathrm{d}t < \infty$, then additionally $G \in C^3(\mathbb{R}^d \setminus \{0\})$ and

$$G(r), |G'(r)|, |G''(r)|, r|G'''(r)| \leqslant S(r), \quad r \geqslant r_0.$$

Theorem (T. Grzywny, ŁL, M. Kassmann)

Let D be an open bounded set. Assume that the measure ν satisfies (A) and (1) and the fundamental solution G satisfies (G). Let $g \in \mathcal{L}(D^c)$ and $f\colon D \mapsto \mathbb{R}$. If $\int_0^1 |G'(t)| t^{d-1} \mathrm{d}t < \infty$, we assume

$$\int_0^1 S(t)\omega_f(t,D) t^{d-1} \mathrm{d}t < \infty, \tag{3}$$

or if $\int_0^1 |G'(t)| t^{d-1} \mathrm{d}t = \infty$, we assume

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Corollary (The fractional Laplace operator $-(-\Delta)^{\alpha/2}$)

Let D be an open bounded set, $g \in \mathcal{L}(D^c)$ and $f : D \mapsto \mathbb{R}$. Suppose that

$$\int_0^1 \omega_f(t, D) t^{\alpha-3} \mathrm{d}t < \infty, \tag{5}$$

for $\alpha \in (0, 1)$, or

$$\int_0^1 \omega_{\nabla f}(t, D) t^{\alpha-2} \mathrm{d}t < \infty. \tag{6}$$

for $\alpha \in [1, 2)$. Then the solution $u \in \mathcal{L}^1(\mathbb{R}^d)$ of the Dirichlet problem (P) belongs to $C^2_{\mathrm{loc}}(D)$ and is unique up to a harmonic function (with respect to $-(-\Delta)^{\alpha/2}$).

Counterexamples for „$\alpha + \beta = 2$"

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Consider a Dirichlet problem

(P1)
$$\begin{aligned} \Delta^{\alpha/2} u &= f \quad \text{in } B_1, \\ u &= 0 \quad \text{in } B_1^c. \end{aligned}$$

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- $f(y) = (y_d)_+ \ln^{-\beta}\left(1 + \left(\frac{1}{y_d}\right)_+\right)$, $\beta \in (0,1)$, for $\alpha = 1$.

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For such functions f, there are weak solutions of (P1) which are not $C^2(B_1)$. In fact, they are even not $C^{1,1}(B_1)$.

Positive examples

1. Let D be an open bounded set. Consider a Dirichlet problem

$$(P2)\qquad \begin{aligned} \Delta^{\alpha/2} u &= f \quad \text{in } D,\\ u &= 0 \quad \text{in } D^c, \end{aligned}$$

where $\alpha \in (0,2)$. It is well known that for any function which is $C^{2-\alpha+\varepsilon}$, $\varepsilon > 0$ (i.e. $\tilde{f}(y) = \left((y_d)_+\right)^{2-\alpha+\varepsilon}$), the solution of (P2) is $C^2_{\mathrm{loc}}(D)$.

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Another positive examples

Any function f satisfying:

- $\omega_{\nabla f}(t,D) \leqslant Ct^{1-\alpha}\ln^{-\beta}\left(1+t^{-1}\right)$, $\beta > 1$, if $\alpha \in (0,1)$,
- $\omega_{\nabla f}(t,D) \leqslant C\ln^{-\beta}\left(1+t^{-1}\right)$, $\beta > 1$, for $\alpha = 1$,
- $\omega_{f}(t,D) \leqslant Ct^{2-\alpha}\ln^{-\beta}\left(1+t^{-1}\right)$, $\beta > 1$, for $\alpha \in (1,2)$.

Positive examples

2. Let $(B_t, t \geqslant 0)$ be a Brownian motion in $\mathbb{R}^d$ and $(S_t, t \geqslant 0)$ — an independent subordinator with the Laplace exponent ϕ. Set $X_t := B_{S_t}$.

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One can show that ν is smooth and (G) is satisfied if $d \geqslant 3$ and $\phi^{-2}\phi'$ satisfies scaling conditions at infinity. Moreover (A) holds with $\nu^* \equiv \nu$ if ϕ is a complete Bernstein function.

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Geometric stable process

Let $\phi(\lambda) = \ln\left(1 + \lambda^{\alpha/2}\right)$, $\alpha \in (0,2)$. Then for any f satisfying $\omega_{\nabla f}(t, D) \leqslant Ct \ln^{1-\varepsilon}(1 + t^{-1})$, $\varepsilon \in (0,1)$, the solution is $C^2(D)$.

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3. Let $d \geqslant 3$, $\alpha \in \left(\frac{3}{2}, 2\right)$, and X_t be a truncated α–stable Lévy process in $\mathbb{R}^d$, i.e. with Lévy measure $\nu(\mathrm{d}x) = |x|^{-d-\alpha}\varphi(x)$, where φ is a cut-off function, i.e. $\varphi \in C^\infty(\mathbb{R}^d)$ and

$$1_{B_{1/2}} \leqslant \varphi \leqslant 1_{B_1}.$$

Then (G), (A) and (1) hold with $\nu^* \equiv 0$. In such case the appropriate $\mathcal{L}^1$ space is simply L^1_{loc}.

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Thank you for your attention!