Singular stochastic integral operators

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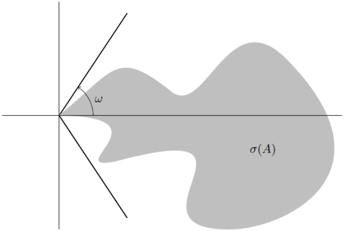
Joint work with Mark Veraar

Overview

Stochastic abstract Cauchy problem

$$\begin{cases} \mathrm{d}u + Au\,\mathrm{d}t = G\,\mathrm{d}W & \text{on } \mathbb{R}_+ \\ u(0) = 0 \end{cases}$$

- $-A$ the generator of an analytic semigroup e^{-tA} on X
  - For example $A = -\Delta$
- X a UMD Banach space
  - For example $X = L^q$ for $q \in (1, \infty)$
- W a standard Brownian motion
- $G \in L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; X)$ for some $p \in (1, \infty)$



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$$u(t) = \int_0^t \mathrm{e}^{-(t-s)A} G(s)\,\mathrm{d}W(s), \qquad t \in \mathbb{R}_+$$

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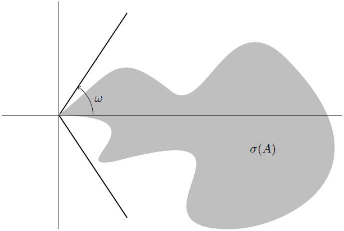

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Stochastic maximal regularity

Definition

We say that A has *stochastic maximal L^p-regularity* if

$$A^{\frac{1}{2}}u \in L^p(\mathbb{R}_+ \times \Omega; X)$$

or in other words, if $S\colon L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; X) \to L^p(\mathbb{R}_+ \times \Omega; X)$ given by

$$SG(t) := \int_0^t A^{\frac{1}{2}} \mathrm{e}^{-(t-s)A} G(s)\,\mathrm{d}W(s), \qquad t \in \mathbb{R}_+$$

is a bounded operator.

Previously studied by:

- Da Prato et al. for $p = 2$ and X Hilbert
- Krylov et al. for $A = -\Delta$ and $X = L^q$ with $2 \leq q \leq p < \infty$
- van Neerven–Veraar–Weis for A with bounded H^∞-calculus for $p \in (2, \infty)$ and $X = L^q$ with $q \in [2, \infty)$

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Stochastic singular integral operators

Goal

Study the p-independence of the L^p-boundedness of

$$S_K G(t) := \int_0^\infty K(t,s)G(s)\,\mathrm{d}W(s), \qquad t \in \mathbb{R}_+$$

where $K\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(X)$ is a singular kernel. Of particular interest is

$$K(t,s) = A^{\frac{1}{2}} \mathrm{e}^{(t-s)A} \mathbf{1}_{t>s}$$

- We call $S_K\colon L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; X) \to L^p(\mathbb{R}_+ \times \Omega; X)$ a *singular stochastic integral operator*
- Deterministic case studied through Calderón–Zygmund theory
- No Calderón–Zygmund theory yet in the stochastic setting

Topic of this talk!

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Stochastic Calderón–Zygmund theory

Stochastic Calderón–Zygmund theorem (L., Veraar '19)

Let X be a UMD Banach space with type 2, $p_0 \in [2,\infty)$ and let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(X)$ satisfy the 2-Hörmander condition. If

$$S_K\colon L^{p_0}_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; X) \to L^{p_0}(\mathbb{R}_+ \times \Omega; X),$$

is bounded, then

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- X has type 2 if for any $x_1, \cdots, x_n \in X$

$$\Big\| \sum_{k=1}^{n} \varepsilon_k x_k \Big\|_{L^2(\Omega;X)} \lesssim \Big(\sum_{k=1}^{n} \|x_k\|^2 \Big)^{1/2}$$

  where $(\varepsilon_k)_{k=1}^n$ is a Rademacher sequence.
- L^q for $q \in [2, \infty)$ is a UMD Banach space with type 2

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- If $K \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(X)$ satisfies

$$\max\Big\{\Big\|\frac{\partial}{\partial t}K(t,s)\Big\|, \Big\|\frac{\partial}{\partial s}K(t,s)\Big\|\Big\} \leq \frac{C}{|t-s|^{3/2}}, \qquad t \neq s$$

then it satisfies the 2-Hörmander condition.

Comments on the proof

- Extrapolation upwards from p_0 to $p \in (p_0, \infty)$
  - Prove BMO-endpoint and interpolate
- Extrapolation downwards from p_0 to $p \in (2, p_0)$
  - Use L^2-Calderón–Zygmund decomposition
  - Cannot exploit mean zero of "bad" part of decomposition
  - Using ideas from Duong–McIntosh '99

Some differences with classical Calderón–Zygmund theory:

- For $X = L^q$ by the Itô isomorphism it is equivalent to consider the operators
$$T_K f(t) := \Big(\int_0^t \underbrace{|K(t,s)f(s)|^2}_{\in L^q} \, ds\Big)^{1/2}, \qquad t \in \mathbb{R}_+$$
for $f \in L^p(\mathbb{R}_+; L^q)$
- The integrals converge absolutely, cancellation takes the form
$$\Big\| s \mapsto \Big(\int_{\mathbb{R}_+} |K(t,s)x|^2 \, ds\Big)^{1/2} \Big\|_{L^q} \lesssim \|x\|_{L^q}, \qquad t \in \mathbb{R}_+, \; x \in L^q$$
- If $X = \mathbb{C}$ and $K(t,s) = k(t-s)$, then S_K is bounded if and only if $k \in L^2(\mathbb{R}_+)$ (analog of $k \in L^1(\mathbb{R}_+)$ in deterministic setting)

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Applications

Corollary

Let X be a UMD Banach space with type 2, $p_0 \in [2, \infty)$ and let $-A$ be the generator of a bounded analytic C_0-semigroup. If A has stochastic maximal L^{p_0}-regularity, then A has maximal L^p-regularity for all $p \in (2, \infty)$.

Other applications include stochastic maximal L^p-regularity for:

- $-\Delta$ on Lebesgue, Besov and Bessel potential spaces
- General A on real interpolation spaces $D_A(\theta, q)$ (Da Prato–Lunardi, Brzeźniak–Hausenblas)
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Weighted Calderón–Zygmund theory

- A locally integrable $w : \mathbb{R}^d \to (0, \infty)$ is called a weight. For $p \in (1, \infty)$ we say $w \in A_p$ if and only if
$$[w]_{A_p} := \sup_{B \subseteq \mathbb{R}^d} \fint_B w \cdot \Big(\fint_B w^{-\frac{1}{p-1}} \Big)^{p-1} < \infty,$$
where the supremum is taken over all balls $B \subseteq \mathbb{R}^d$.
- The space $L^p(\mathbb{R}^d, w; X)$ consist of all strongly measurable functions $f : \mathbb{R}^d \to X$ such that
$$\|f\|_{L^p(\mathbb{R}^d,w;X)} := \Big(\int_{\mathbb{R}^d} \|f(t)\|_X^p w(t) \,\mathrm{d}t \Big)^{\frac{1}{p}} < \infty$$

Deterministic A_2-theorem (Hytönen '12)

Let T be a singular integral operator with a standard kernel, then for all $w \in A_p$
$$\|T\|_{L^p(\mathbb{R}^d,w) \to L^p(\mathbb{R}^d,w)} \lesssim [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}$$

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Sparse domination (Lerner, Ombrosi '19)

Take $p_0, p_1 \in [1,\infty)$ and set $p := \max\{p_0, p_1\}$. Suppose that

- T is a bounded sublinear operator from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0,\infty}(\mathbb{R}^d)$
- $\mathcal{M}_T^{\#,\alpha}$ is bounded from $L^{p_1}(\mathbb{R}^d)$ to $L^{p_1,\infty}(\mathbb{R}^d)$ for some $\alpha \geq 3$.

Then for any compactly supported $f \in L^p(\mathbb{R}^d)$ there exists an η-sparse collection of cubes $\mathcal{S}$ such that for a.e. $t \in \mathbb{R}^d$

$$|Tf(t)| \lesssim \sum_{Q\in\mathcal{S}} \left(\fint_Q |f|^p \right)^{1/p} \mathbf{1}_Q(t).$$

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Here $\mathcal{M}_T^{\#,\alpha}$ is the grand maximal truncation operator given by

$$\mathcal{M}_T^{\#,\alpha} f(t) := \sup_{Q\ni t} \operatorname*{ess\,sup}_{t',t''\in Q} |T(f\,\mathbf{1}_{\mathbb{R}^d\setminus\alpha Q})(t') - T(f\,\mathbf{1}_{\mathbb{R}^d\setminus\alpha Q})(t'')|, \qquad t \in \mathbb{R}^d$$

in which the supremum is taken over all cubes Q containing t.

Vector-valued sparse domination (L. '19)

Let X and Y be Banach spaces.
Take $p_0, p_1 \in [1, \infty)$ and set $p := \max\{p_0, p_1\}$. Suppose that

- T is a bounded sublinear operator from $L^{p_0}(\mathbb{R}^d; X)$ to $L^{p_0,\infty}(\mathbb{R}^d; Y)$
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Then for any compactly supported $f \in L^p(\mathbb{R}^d; X)$ there exists an η-sparse collection of cubes $\mathcal{S}$ such that for a.e. $t \in \mathbb{R}^d$

$$\|Tf(t)\|_Y \lesssim \sum_{Q \in \mathcal{S}} \left(\fint_Q \|f\|_X^p \right)^{1/p} \mathbf{1}_Q(t).$$

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Vector-valued sparse domination in a SHT (L. '19)

Let X and Y be Banach spaces and (S, d, μ) a space of homogeneous type. Take $p_0, p_1 \in [1, \infty)$ and set $p := \max\{p_0, p_1\}$. Suppose that

- T is a bounded sublinear operator from $L^{p_0}(S; X)$ to $L^{p_0,\infty}(S; Y)$
- $\mathcal{M}_T^{\#,\alpha}$ is bounded from $L^{p_1}(S; X)$ to $L^{p_1,\infty}(S)$ for some $\alpha \geq 3/\delta$.

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Vector-valued ℓ^r-sparse domination in a SHT (L. '19)

Let X and Y be Banach spaces and (S, d, μ) a space of homogeneous type. Take $p_0, p_1, r \in [1, \infty)$ and set $p := \max\{p_0, p_1\}$. Suppose that

- T is a bounded sublinear operator from $L^{p_0}(S; X)$ to $L^{p_0,\infty}(S; Y)$
- $\mathcal{M}_T^{\#,\alpha}$ is bounded from $L^{p_1}(S; X)$ to $L^{p_1,\infty}(S)$ for some $\alpha \geq 3$.
- There is a constant $C > 0$ such that for any disjointly supported $f_1, f_2 \in L^p(S; X)$

$$\|T(f_1 + f_2)(t)\|_Y^r \leq \|Tf_1(t)\|_Y^r + C\,\|Tf_2(t)\|_Y^r, \qquad t \in S.$$

Then for any compactly supported $f \in L^p(S; X)$ there exists an η-sparse collection of cubes $\mathcal{S}$ such that for a.e. $t \in S$

$$\|Tf(t)\|_Y \lesssim \Big(\sum_{Q \in \mathcal{S}} \Big(\fint_Q \|f\|_X^p\Big)^{r/p} \mathbf{1}_Q(t)\Big)^{1/r}.$$

Here $\mathcal{M}_T^{\#,\alpha}$ is the grand maximal truncation operator given by

$$\mathcal{M}_T^{\#,\alpha} f(t) := \sup_{Q \ni t} \operatorname*{ess\,sup}_{t', t'' \in Q} \|T(f\,\mathbf{1}_{S \setminus \alpha Q})(t') - T(f\,\mathbf{1}_{S \setminus \alpha Q})(t'')\|_Y, \qquad t \in S$$

in which the supremum is taken over all cubes Q containing t.

Stochastic "A_2-theorem" (L., Veraar '19)

Let X be a UMD Banach space with type 2, $p_0 \in [2, \infty)$ and let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(X)$ satisfy the 2-Dini condition. If

$$S_K\colon L^{p_0}_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; X) \to L^{p_0}(\mathbb{R}_+ \times \Omega; X),$$

is bounded, then

$$S_K\colon L^{p}_{\mathcal{F}}(\mathbb{R}_+ \times \Omega, w; X) \to L^{p}(\mathbb{R}_+ \times \Omega, w; X)$$

is bounded for all $p \in (2, \infty)$ and all $w \in A_{p/2}(\mathbb{R}_+)$. In particular

$$\|S_K\| \lesssim [w]_{A_{p/2}(\mathbb{R}_+)}^{\max\{\frac{1}{2}, \frac{1}{p-2}\}}.$$

If $K\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{L}(X)$ satisfies

$$\max\Big\{\Big\|\frac{\partial}{\partial s}K(t,s)\Big\|, \Big\|\frac{\partial}{\partial t}K(t,s)\Big\|\Big\} \leq \frac{C}{|t-s|^{3/2}}, \qquad t \neq s$$

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Comments on the proof

- Apply sparse domination with $p_0 = p_1 = r = 2$.
- T_K is actually weak L^2-bounded by unweighted theory
- For $f \in L^2(\mathbb{R}_+; X)$ we have by the 2-Dini condition

$$\mathcal{M}^{\#,\alpha}_{T_K} f(t) \leq M_2(\|f\|_X)(t), \qquad t \in \mathbb{R}_+$$

  and M_2 is weak L^2-bounded.
- The 2-sublinearity of T_K is implied by

$$\frac{\|x_1 + x_2\|_X^2}{2} + \frac{\|x_1 - x_2\|_X^2}{2} \leq \|x_1\|_X^2 + C\,\|x_2\|_X^2, \qquad x_1, x_2 \in X.$$

  which is equivalent (up to renorming) to martingale type 2 of X.
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$$f(t) \mapsto \Big(\sum_{Q \in \mathcal{S}} \Big(\fint_Q |f|^2\Big)\Big)^{1/2} \mathbf{1}_Q(t).$$

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Sharpness of the weighted estimate

- The stochastic singular integral operator S_K with $K(t,s) = \frac{1}{(t+s)^{1/2}}$ is related to the Hilbert operator

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which implies

$$\|S_K\|_{L^p(\mathbb{R}_+)\to L^p(\mathbb{R}_+)} \simeq \frac{1}{\sin(2\pi/p)^{1/2}},$$

with higher dimensional analogs by Osękowski '17.

- By a result of Luque–Pérez–Rela '15 (extended by Frey–Nieraeth '19) this implies that if

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Applications

- All previous examples admit weights $w \in A_{p/2}(\mathbb{R}_+)$
- In particular the power weights $w(t) = t^\alpha$ for $\alpha \in (-1, p/2 - 1)$ can be used to allow for rough initial data

Other applications of the sparse domination theorem:

- Vector-valued Littlewood–Paley–Rubio de Francia estimates (Potapov–Sukochev–Xu '12)
- Deterministic vector–valued A_2-theorem in spaces of homogeneous type (Hänninen–Hytönen '14), (Nazarov–Resnikov–Volberg '13)
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Thank you for your attention!