Probability and Analysis 2019
Będlewo, May 20th, 2019

Critical phenomena in random discrete structures

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Typically, we draw the vertices of G as points and the edges of G are represented by line segments.

AND MORE SPECIFICALLY, ON RANDOM GRAPHS

DEFINITION OF $G(n, p)$

$G(n, p)$ is a random graph with vertex set $\{1, 2, \ldots, n\}$ in which each edge is generated with probability p, independently for each of $\binom{n}{2}$ pairs.
More specifically, $G(n, p)$ is probability space, where

$$\mathbb{P}(G(n, p) = G) = \binom{\binom{n}{2}}{|E(G)|} p^{|E(G)|} (1 - p)^{\binom{n}{2} - |E(G)|} .$$

RANDOM PROCESS $\{G(n, p) : 0 \leq p \leq 1\}$

Equivalently, for each pair of vertices ij we can generate a random variable U_{ij} with uniform distribution in $[0, 1]$ and define the set of edges of $G(n, p)$ as

$$E = \{ij \; : \; U_{ij} \leq p\} .$$

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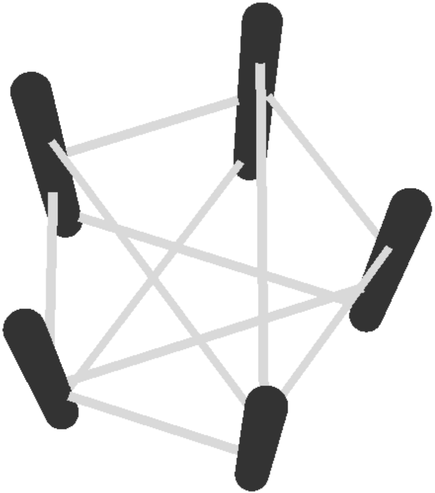

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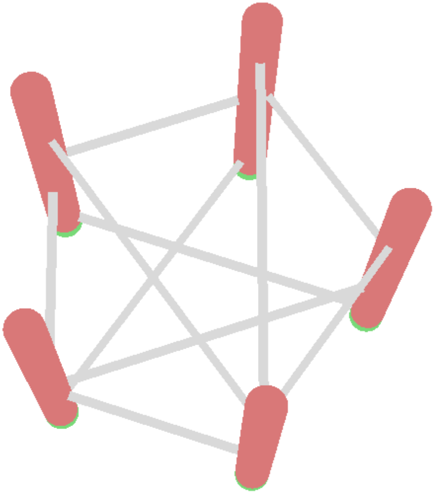

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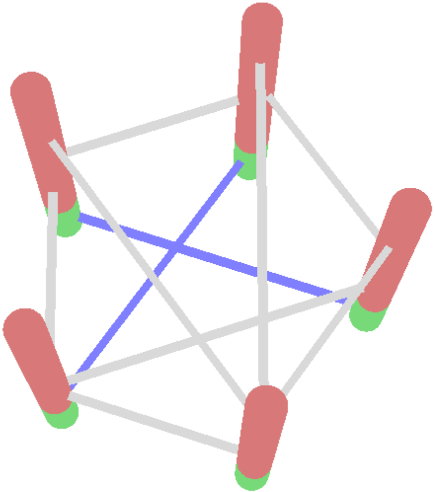

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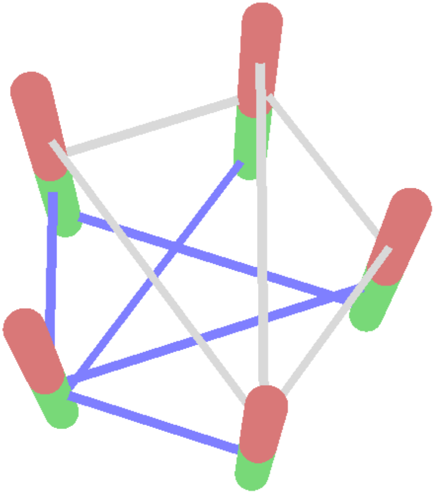

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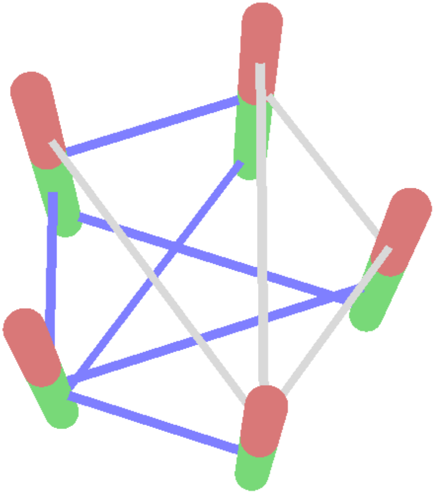

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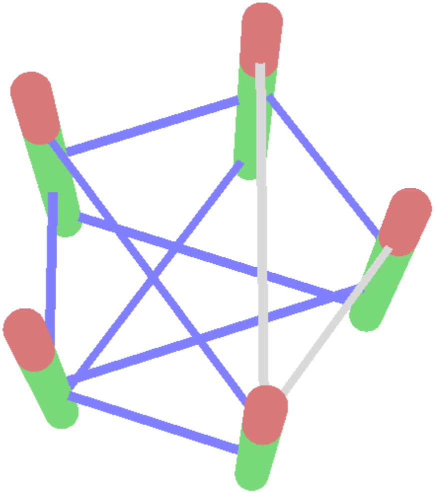

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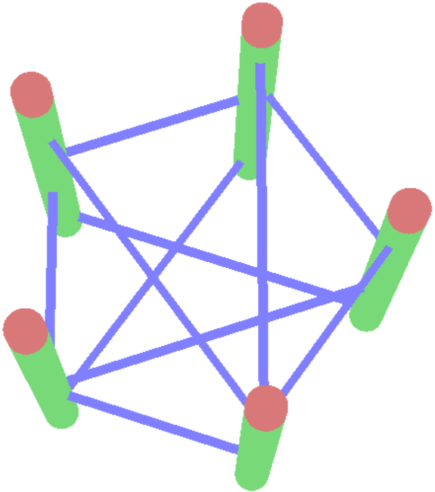

A USEFUL OBSERVATION

OBSERVATION

From the process $\{G(n,p) : 0 \leq p \leq 1\}$ we get a natural coupling which shows that

$$G(n,p_1) \subseteq G(n,p_2),$$

whenever $p_1 \leq p_2$.

n IS ALWAYS FINITE BUT VERY LARGE

In random graph theory we are interested mainly in typical properties of $G(n, p)$.

For a given function $p = p(n)$ (e.g. $p = 3/n$) we say that $G(n, p)$ has some property $\mathcal{A}$ asymptotically almost surely (or, briefly, aas) if the probability that $G(n, p)$ has $\mathcal{A}$ tends to 1 as $n \to \infty$.

Erdős, Rényi seminal paper (1960)

Theorem Erdős, Rényi'60

If $np \to c > 0$, then $\mathbb{P}(G(n,p) \not\supset K_3) = \exp(-c^3/6)$.

Theorem Erdős, Rényi'60

Let $L_1(n,p)$ be the size of the largest component of $G(n,p)$.

(i) If $np \to c < 1$, then aas $L_1(n,p) = \Theta(\log n)$.

(ii) If $np \to c > 1$, then aas $L_1(n,p) = \Theta(n)$.

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The phase transition

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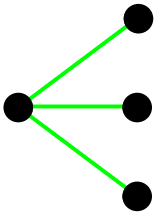

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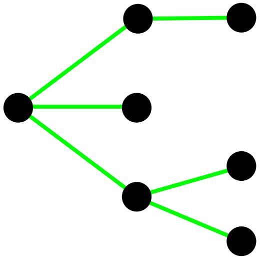

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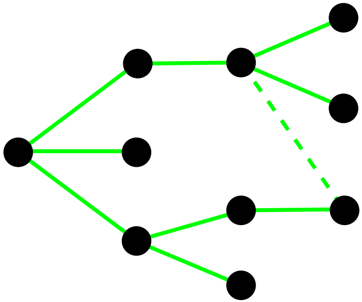

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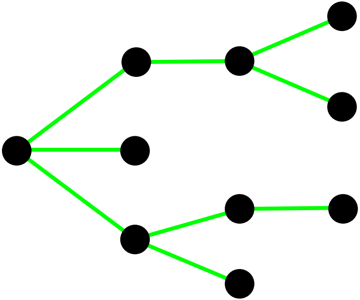

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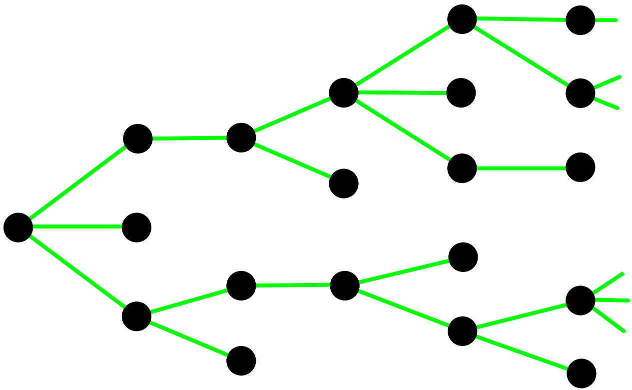

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Theorem Bollobás'84; Łuczak'90

Let $\omega(n) \to \infty$ and $\omega(n) = o(n^{1/3})$.

(I) If $np = 1 - \omega n^{-1/3}$, then aas $L_1(n,p) = \Theta\left(\frac{n^{2/3}}{\omega^2}\log\omega\right)$.

(II) If $np = 1 + \Theta(n^{-1/3})$, then aas $L_1(n,p) = \Theta(n^{2/3})$.

(III) If $np = 1 + \omega n^{-1/3}$, then aas $L_1(n,p) = (2 + o(1))\omega n^{2/3}$.

The phase transition

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The width of the phase transition in $G(n, p)$ is $n^{-1/3}$.

Janson, Knuth, Łuczak, Pittel'93
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Local limit theorems for the sizes of largest components in the critical window.

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The random graph process as the 'race of components'.

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Why some 'thresholds' are 'coarse' while others are 'sharp'?

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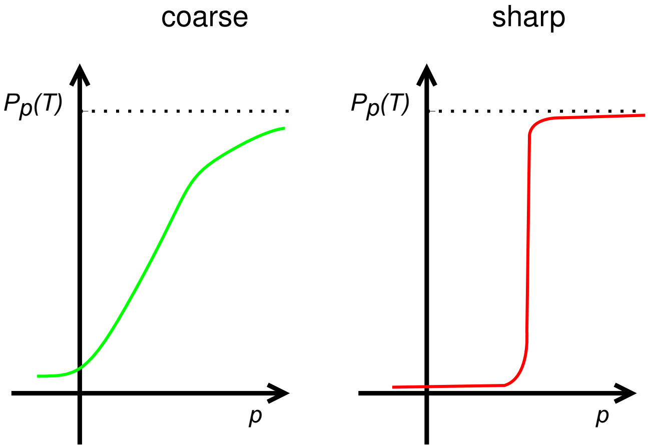
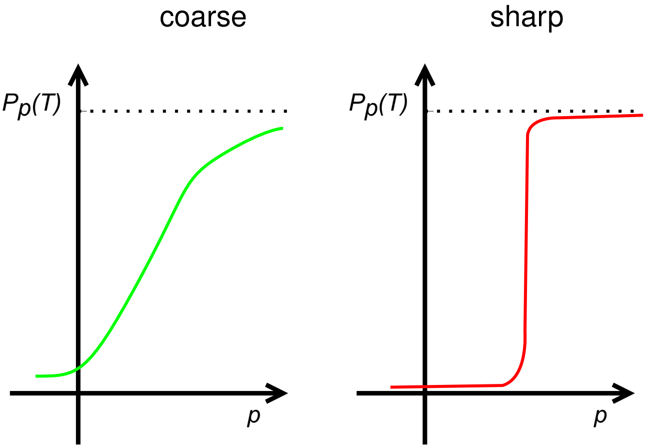
TWO TYPES OF THRESHOLDS



Two types of thresholds



Thus, for instance, the threshold for the property that a graph contains a triangle is coarse in $G(n, p)$.

General theory of (sharp) thresholds

Kahn, Kalai, Linial'88

$\Downarrow$

Bourgain, Kahn, Kalai, Katznelson, Linial'92

$\Downarrow$

Friedgut+Bourgain'99

General theory of (sharp) thresholds

Suppose a random subset $\mathcal{R}_p$ of a set Ω is obtained choosing elements of Ω independently at random with probability p.
Let A be an increasing property of subsets of Ω.

Theorem Friedgut+Bourgain'99

A property A has a coarse threshold if and only if it is 'local'.

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Local and non-local properties

Local properties

If $np \to c > 0$, then $\mathbb{P}(G(n,p) \not\supset K_3) = \exp(-(c^3/6))$.

Non-Local properties

Let $L_1(n,p)$ be the size of the largest component of $G(n,p)$.

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Random groups

Quote

I feel, random groups altogether may grow up as healthy as random graphs, for example.

Misha Gromov *Spaces and questions* 1999

Group presentations

$$G = \langle S | R \rangle$$

is a group which consists of words with letters $a, b, \ldots$ (as well as its formal inverses $a^{-1}, b^{-1}, \ldots$) from an alphabet S in which we can cancel all words from set R.

Example

In the group

$$G = \langle \{a, b\} | aba^{-1}b^{-1} \rangle$$

we have $aba^{-1}b^{-1} = e$, i.e.

$$ab = aba^{-1}b^{-1}ba = aba^{-1}b^{-1}ba = ba\,,$$

so

$$G = \{a^n b^m : a, b \in \mathbb{Z}\} = \mathbb{Z}^2\,.$$

Finitely presented groups are often hard to study

Presentations are sometimes hard to deal with, both in theory

Theorem

Given presentation $\langle S|R\rangle$ of a group Γ it is undecidable if a given word is equivalent to 0 in Γ.

Many properties of groups with natural short finite presentations are unkown (e.g. it is not known if Thompson group F is amenable).

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RANDOM GROUP $\Gamma(n, p)$

DEFINITION GROMOV'88; ŻUK'03

$$\Gamma(n, p) = \langle \{g_1, g_2, \ldots, g_n\} | \mathcal{R}_p \rangle$$

where each relation of length three belongs to $\mathcal{R}_p$ independently with probability p.

THE EVOLUTION OF $\Gamma(n, p)$

THEOREM ŻUK'03

For every constant $\epsilon > 0$ the following holds.

- If $p \leq n^{-2-\epsilon}$ then aas $\Gamma(n, p)$ is free.
- If $n^{-2+\epsilon} \leq p \leq n^{-3/2-\epsilon}$, then aas $\Gamma(n, p)$ is infinite, hyperbolic, and has Kazdhan's property (T).
- If $p \geq n^{-3/2+\epsilon}$, then aas $\Gamma(n, p)$ is trivial.

Collapsing $\Gamma(n, p)$

Theorem Żuk'03

Let $\epsilon > 0$. Then

- If $p \leq n^{-3/2-\epsilon}$, then aas $\Gamma(n, p)$ is infinite.
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Theorem Antoniuk, Łuczak, Świątkowski'14

There exists a constant $c > 0$ such that if $p \geq cn^{-3/2}$, then aas $\Gamma(n, p)$ is trivial.

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Conjecture Antoniuk, Łuczak, Świątkowski'14

There exists a constant $c' > 0$ such that if $p \leq c'n^{-3/2}$, then aas $\Gamma(n, p)$ is infinite (and hyperbolic).

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- If $p \geq n^{-3/2+\epsilon}$, then aas $\Gamma(n, p)$ is trivial.

Theorem Antoniuk, Friedgut, Łuczak'17

There exists a function $c(n)$ such that for every $\epsilon > 0$ the following holds.

- If $p \geq (1+\epsilon)c(n)n^{-3/2}$, then aas $\Gamma(n, p)$ is trivial.
- If $p \leq (1-\epsilon)c(n)n^{-3/2}$, then aas $\Gamma(n, p)$ is not trivial.

Conjecture Antoniuk, Friedgut, Łuczak'17

$c(n) \to c > 0$ as $n \to \infty$.

Collapsing $\Gamma(n, p)$

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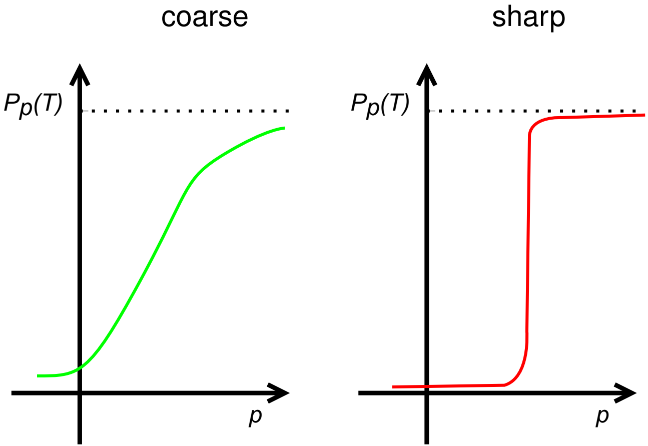
Back to the two types of thresholds



We claim that the threshold for collapsing is sharp.

Friedgut-Bourgain Theorem

Suppose a random subset $\mathcal{R}_p$ of a set Ω is obtained choosing elements of Ω independently at random with probability p.
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Theorem Friedgut+Bourgain'99

A property A has a coarse threshold if and only if it is 'local'.

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Example

Consider the following properties of $\Gamma(n,p) = \langle S | \mathcal{R}(n,p) \rangle$
A_1: five generators of $\Gamma(n,p)$ are equivalent to the identity,
A_2: all generators of $\Gamma(n,p)$ are equivalent to the identity.

Then, A_1 has a coarse threshold, while, as we see shortly, the threshold for A_2 is sharp.

Sharp threshold for the collapse

Theorem Friedgut+Bourgain'99

A property A has a coarse threshold if and only if it is 'local'.

Theorem Antoniuk, Friedgut, Łuczak'17

The threshold for collapsing $\Gamma(n, p)$ which occurs for $p \sim n^{-3/2+o(1)}$ is sharp.

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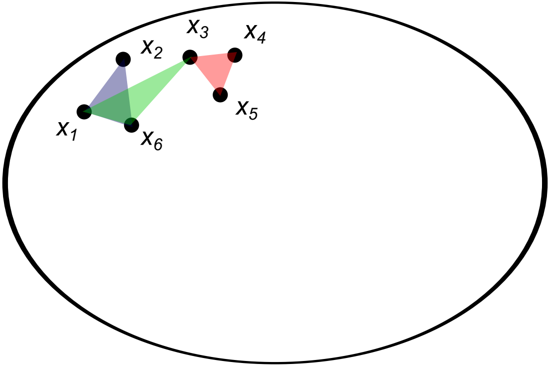
Proof We have to show that collapsing is not 'local', i.e. adding a few relations to $\Gamma(n, p)$ does not change the probability of collapsing more than changing probability p to $(1 + \epsilon)p$, for some $\epsilon > 0$.

The 'local' graph



$x_1x_2x_6 = e$ & $x_3x_5x_4 = e$ & $x_1x_3x_6 = e$

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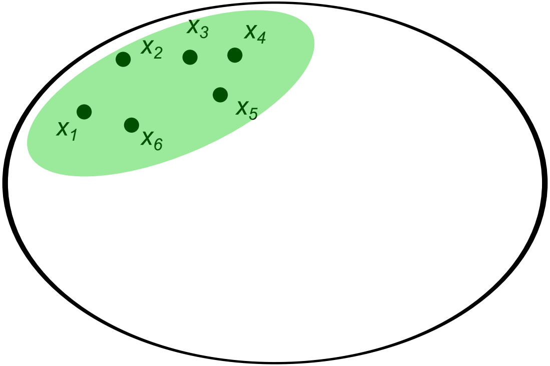

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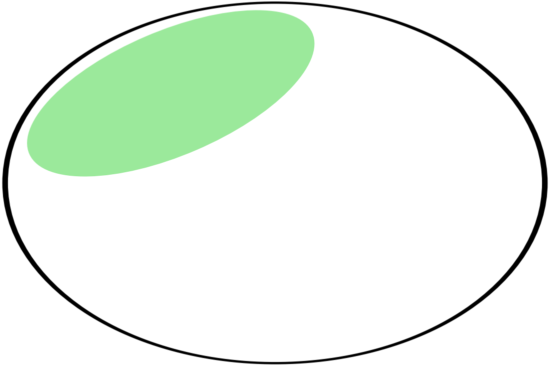

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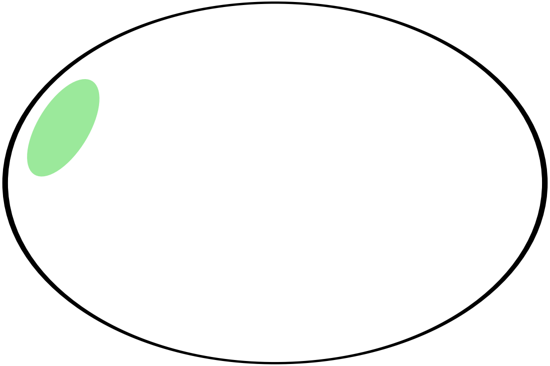

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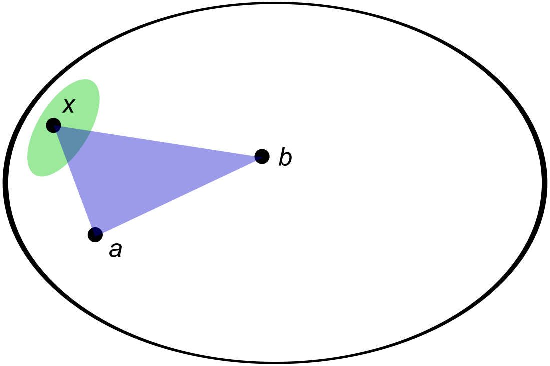

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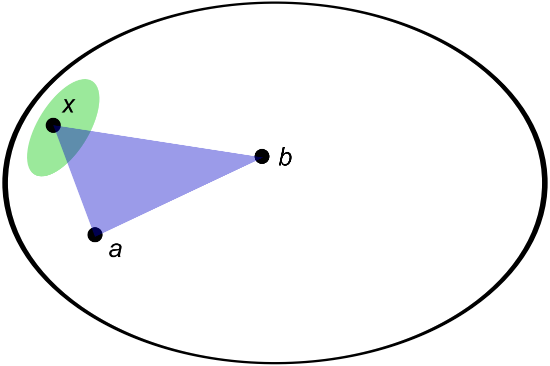

$xab = e \implies ab = e \implies a = b^{-1}$

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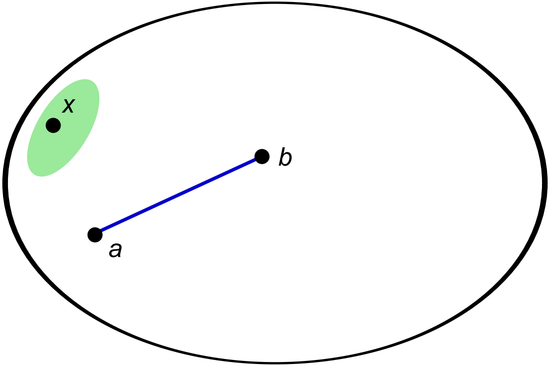

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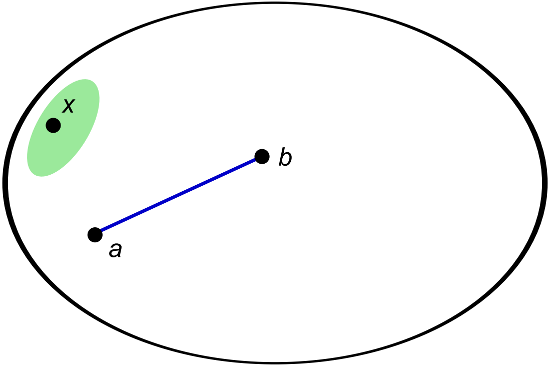

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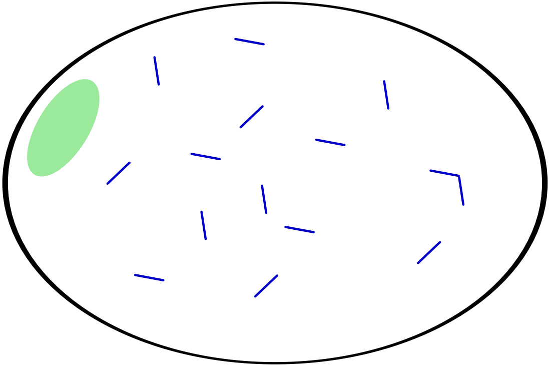
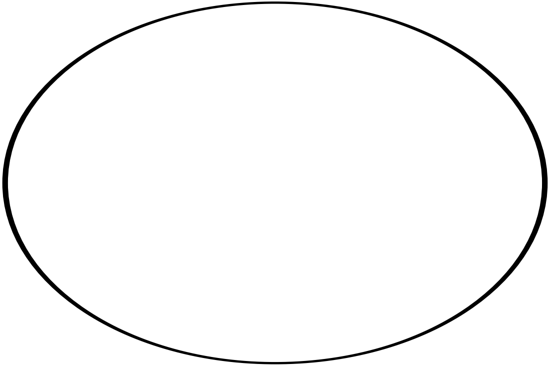
$\rho_1 = \Theta(p) = n^{-3/2+o(1)}$

The blue 'local' graph



$\rho_1 = \Theta(p) = n^{-3/2+o(1)}$

The ‘global’ graph



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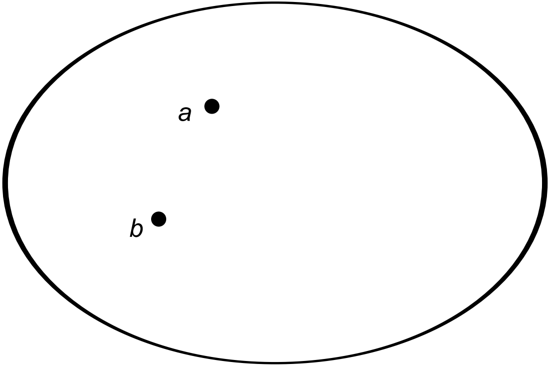
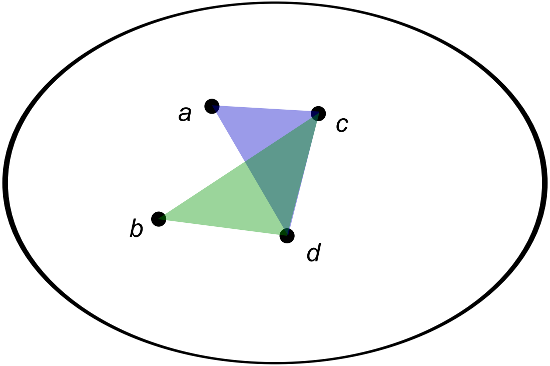


The 'global' graph



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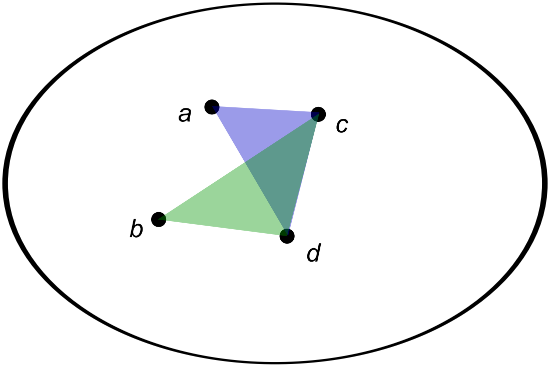

$acd = e$ & $b^{-1}cd = e$

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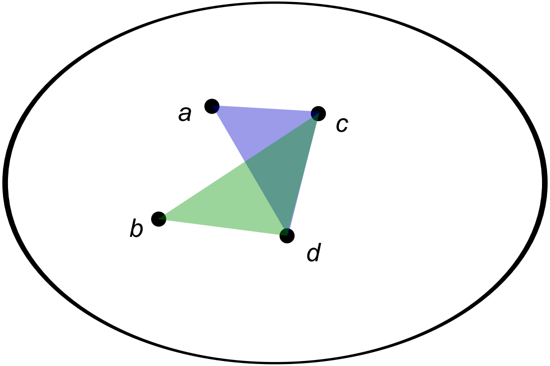

$acd = e$ & $b^{-1}cd = e \implies a = b^{-1}$

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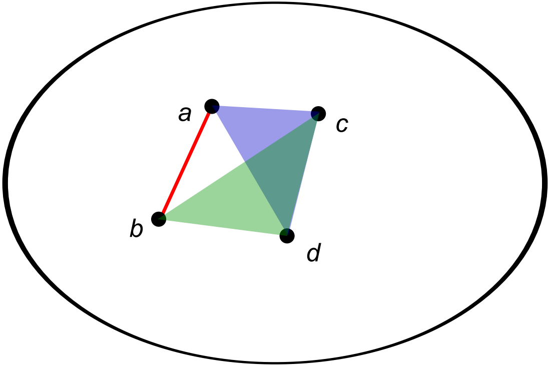

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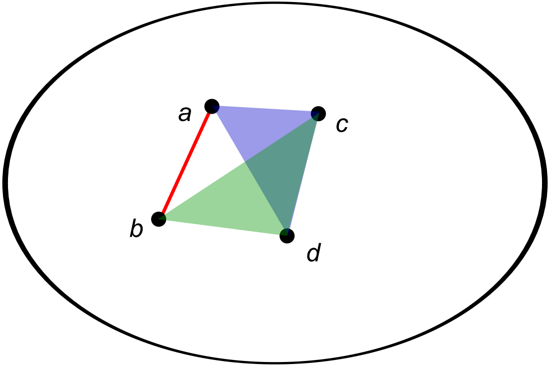

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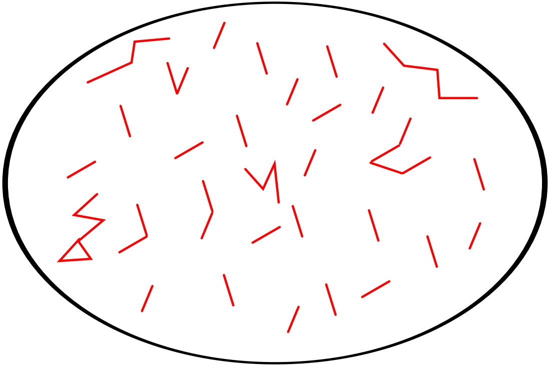
$\rho_2 = \Theta(n^2(\epsilon p)^2) = n^{-1+o(1)}$

The red 'global' graph



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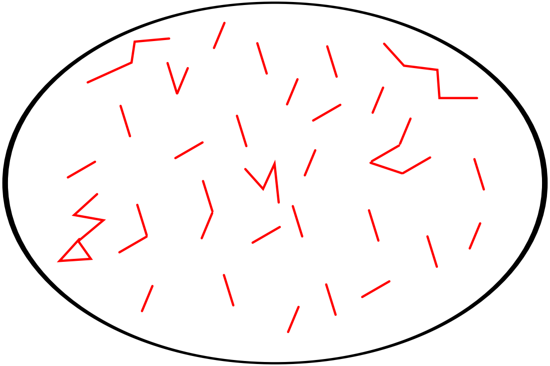
THE RED 'GLOBAL' GRAPH



$a = b^{-1}$

$\rho_2 = \Theta(n^2(\epsilon p)^2) = n^{-1+o(1)} \gg \rho_1 = \Theta(p) = n^{-3/2+o(1)}$ QED

THEOREM Żuk'03

For every constant $\epsilon > 0$ the following holds.

- If $p \leq n^{-2-\epsilon}$ then aas $\Gamma(n, p)$ is free.
- If $n^{-2+\epsilon} \leq p \leq n^{-3/2-\epsilon}$, then aas $\Gamma(n, p)$ is infinite, hyperbolic, and has Kazdhan's property (T).
- If $p \geq n^{-3/2+\epsilon}$, then aas $\Gamma(n, p)$ is trivial.

The evolution of the random group

Theorem Żuk'03

Let $\epsilon > 0$.

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Theorem Antoniuk, Łuczak, Świątkowski'14; Antoniuk, Łuczak, Prytuła, Przytycki 19+

There exists an (explicit) constant $c > 0$ such that every $\epsilon > 0$:

- if $p \leq (c-\epsilon)n^{-2}$ then aas $\Gamma(n,p)$ is free.
- If $p \geq (c+\epsilon)n^{-2}$, then aas $\Gamma(n,p)$ is not free.

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i.e. from the 'mean-field approximation' to finite dimensions

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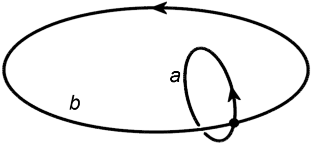
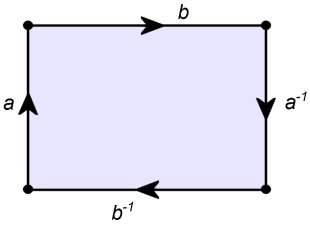
Limit graphs and flag-algebras

From random to pseudo-random structures

The presentation complex

$$\mathbb{Z}^2 = \langle \{a, b\} | aba^{-1}b^{-1} \rangle$$





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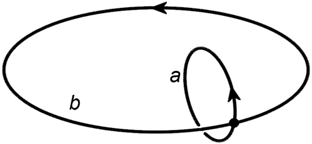

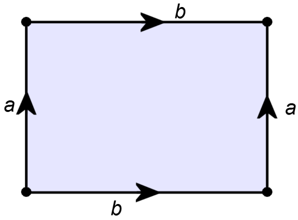

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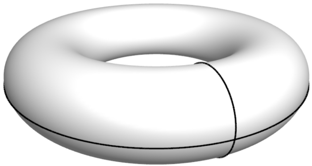
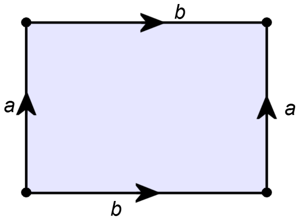
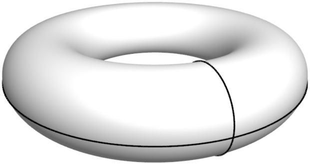
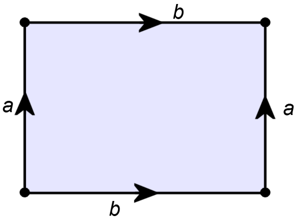




THE PRESENTATION COMPLEX

$\mathbb{Z}^2 = \langle \{a, b\} | aba^{-1}b^{-1} \rangle = \pi_1($  $)$





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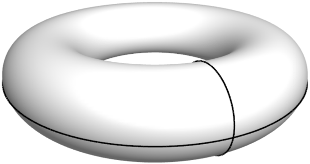

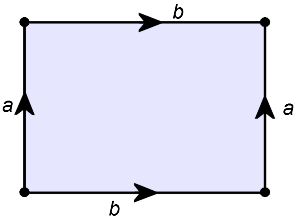

FINITELY PRESENTED GROUPS ARE '2-DIMENSIONAL'

Thus,

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and, in general, each finitely presented groups can be viewed as the fundamental group of its (2-dimensional) presentation complex.

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What about random groups of higher dimensions?

Thank you!