Tracy-Widom fluctuations in 2D random Schroedinger operators

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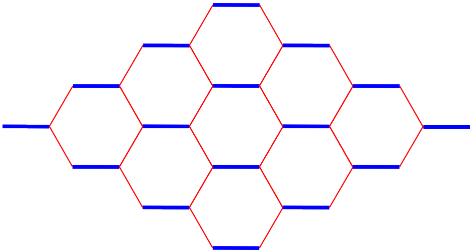
Setup

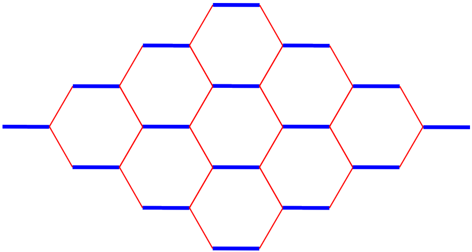


Consider a **random Schroedinger operator** H_n acting on a function f on the hexagonal lattice of size n:

$$H_n f(u) = \sum_{u \sim v} w_{u,v} f(v)$$

where $w_{u,v}$ - (random) **edge weight** between u and v.
Red edges – weight 1. Blue edges – i.i.d weights drawn from some distribution.

Setup



Goal: say something about the eigenvalues of H_n as $n \to \infty$.

E.g. what can be said about the limiting distribution of λ_n, the smallest positive eigenvalue of H_n? Mean size, fluctuations?

Main result

We take blue weights from the Gamma distribution $\Gamma(\gamma)$ with some $\gamma > 0$.

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Main theorem

The smallest positive eigenvalue λ_n of H_n satisfies for small enough γ:

$$\mathbb{P}\left(\frac{-\log \lambda_n - f_\gamma n}{n^{1/3}} \leq r\right) \to F_{GUE}\left(\left(\frac{g_\gamma}{2}\right)^{-1/3} r\right)$$

where f_γ, g_γ are explicit constanst and F_{GUE} is the **Tracy-Widom distribution function**.

- so for large n we have $\lambda_n \approx \exp\left(-f_\gamma n - \chi n^{1/3}\right)$, where χ has Tracy-Widom distribution

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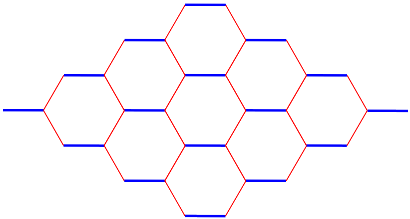
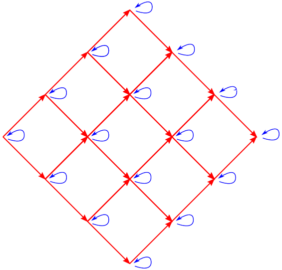
- connections to **directed polymers**

Proof

Simple linear algebra shows that the eigenvalues of H_n are equal in absolute value to the **singular values** of the adjacency matrix A_n of the directed graph shown to the right:



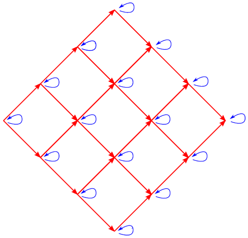


where blue loops have random Gamma weights.

$$(A_n f)(u) = w_u f(u) + \sum_{v \to u} f(v)$$

Proof

How to handle singular values of A_n near zero?

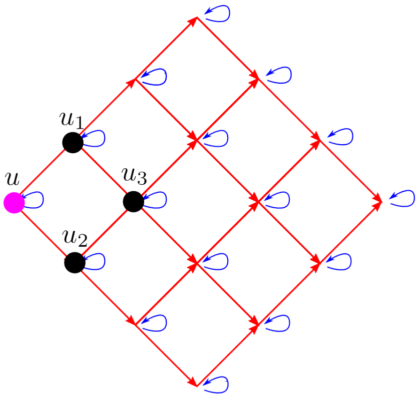


Let us pick a vertex u and try to solve for an *almost zero eigenvector* f_u:

$$A_n f_u = \delta_u$$

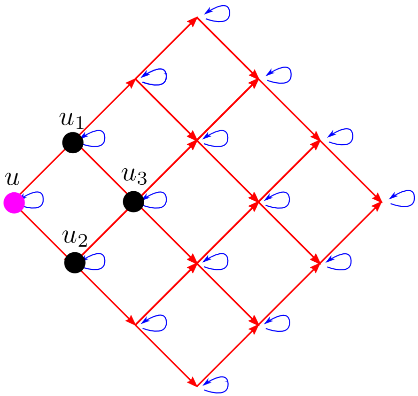
where $\delta(u) = 1$ at u and 0 at other vertices.

Proof



$$A_n f_u(u) = w_u f_u(u) = 1 \Longrightarrow f_u(u) = \frac{1}{w_u}$$

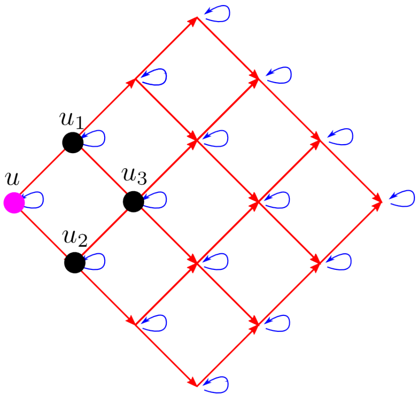
Proof



$$A_n f_u(u) = w_u f_u(u) = 1 \Longrightarrow f_u(u) = \frac{1}{w_u}$$

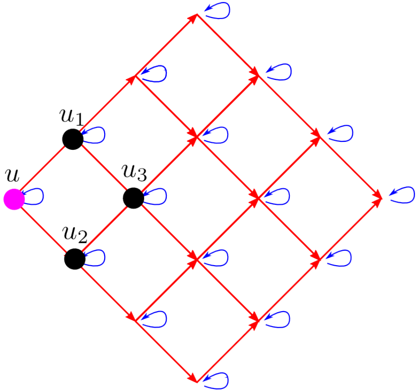
$$A_n f_u(u_1) = w_{u_1} f_u(u_1) + f_u(u) = 0 \Longrightarrow f_u(u_1) = \frac{-1}{w_u w_{u_1}}$$

Proof



$$f_u(u_2) = \frac{-1}{w_u w_{u_2}}$$

Proof



$$f_u(u_2) = \frac{-1}{w_u w_{u_2}}$$

$$f_u(u_3) = \frac{1}{w_u w_{u_1} w_{u_3}} + \frac{1}{w_u w_{u_2} w_{u_3}}$$

Proof

We see that in general the solution to $A_n f = \delta_u$ will satisfy:

$$f_u(v) = \pm \sum_{\pi : u \to v} \prod_{x \in \pi} \frac{1}{w_x}$$

where the sum is over all paths π from u to v.

Proof

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where the sum is over all paths π from u to v.
This is the same as the **partition function** of the **directed Gamma polymer** on the square lattice! (Seppalainen)

Directed polymer model

Directed Gamma polymer on the square lattice:

- vertices equipped with random **inverse Gamma** weights w_u
- up-right paths from $(1,1)$ to (n,n) equipped with weights:

$$\mathrm{wt}(\pi) = \prod_{u \in \pi} w_u$$

- consider the **partition function**:

$$Z_n = \sum_{\pi} \mathrm{wt}(\pi)$$

What can be said about the behavior of Z_n as $n \to \infty$?

Directed polymer model

Theorem (Borodin-Corwin-Remenik '13, Krishnan-Quastel '16)

The partition function of the log Gamma polymer satisfies:

$$\mathbb{P}\left(\frac{\log Z_n - f_\gamma n}{n^{1/3}} \leq r\right) \to F_{GUE}\left(\left(\frac{g_\gamma}{2}\right)^{-1/3} r\right)$$

where f_γ, g_γ are explicit constanst and F_{GUE} is the **Tracy-Widom distribution function**.

- proof relies on combinatorics related to the *KPZ universality class*

Proof

Back to our lattice...

Since $A_n f_u = \delta_u$, the matrix A_n^{-1} has entries $f_u(v)$ (= partition function from u to v). Easy to see that:

$$\max_{u,v} f_u(v) \leq \sigma_1(A_n^{-1}) \leq n^2 \max_{u,v} f_u(v)$$

In our case $\max_{u,v} f_u(v)$ is maximized when $u = (1,1), v = (n,n)$.

So basically the largest singular value of A_n^{-1} (= the smallest singular value of A_n) behaves like the partition function Z_n! (up to small error) $\Longrightarrow$ has Tracy-Widom fluctuations.

Open questions

- intermediate disorder?
- what for general distributions? (Gamma is very specific – allows explicit formulae)
- what for i.i.d. weights on all edges? (blue and red)
- what about the k'th eigenvalue? Conjecture – the same as k top eigenvalues of GUE random matrix, related to more general type of *non-intersecting* partition functions