Extended Bernoulli comparison

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Outline

Canonical Bernoulli process

- Consider a subset T of ℓ^2
- i.i.d. Rademacher sequence $\varepsilon_1, \varepsilon_2, \ldots$ i.e. $\mathbb{P}(\varepsilon_i = \pm 1) = \frac{1}{2}$
  For $t = (t_i) \in T$ we define a random variable

$$B_t = \sum_{i=1}^{\infty} t_i \varepsilon_i.$$

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- i.i.d. sequence $g_1, g_2, \dots \sim \mathcal{N}(0,1)$

$$G_t = \sum_{i=1}^{\infty} t_i g_i.$$

Canonical Bernoulli process

For $B_t = \sum_{i=1}^{\infty} t_i \varepsilon_i$ define

$$S_B(T) = \mathbb{E} \sup_{t \in T} B_t$$

and for $G_t = \sum_{i=1}^{\infty} t_i g_i$

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Theorem (Bednorz, Latała, Ann.of Math., 2014)

For any set $T \subset \ell^2$ with $S_B(T) \leq \infty$ there exist T_1 and T_2 such that $T \subset T_1 + T_2$ and for some universal constant L we have

$$S_B(T) \geq L^{-1}(\sup_{t \in T_1} \|t\|_1 + S_G(T_2)).$$

Contraction property

Suppose that $\varphi : T \to \ell^2$ is such that for any $p \geq 2$ and $s, t \in T$ we have that

$$\|B_{\varphi(t)} - B_{\varphi(s)}\|_p \leq \|B_t - B_s\|_p.$$

(For a random variable ξ and $p > 0$ we put $\|\xi\|_p = (\mathbb{E}|\xi|^p)^{\frac{1}{p}}$)
Does it imply that

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Does it imply that

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- For G_t it is enough to have moment comparison for $p = 2$
- For B_t there is a counterexample for $p = 2$

Contraction property

- For G_t we have Slepian's lemma, which states that if

$$\|G_{\varphi(t)} - G_{\varphi(s)}\|_2 \leq \|G_t - G_s\|_2$$

then

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- For B_t we can consider

$$T = \{e_1, e_2, \dots\},$$

where e_i are elements of the basis in ℓ^1 and

$$\varphi(T) = \{\frac{1}{\sqrt{n}}(\underbrace{\pm 1, \pm 1, \dots, \pm 1}_{n \text{ terms}}, 0, 0, \dots)\}.$$

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Then,

$$\|B_{\varphi(t)} - B_{\varphi(s)}\|_2 = \sqrt{2} = \|B_t - B_s\|_2,$$

$S_B(T) = 1$, but $S_B(\varphi(T)) \geq \sqrt{n}$.

Contraction property

Talagrand (1993): Consider contractions $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i \geq 1$, with $\varphi_i(0) = 0$ (by the contraction we mean that $|\varphi_i(x) - \varphi_i(y)| \leq |x - y|$). Then,

$$S_B(\varphi(T)) \leq S_B(T),$$

where $S_B(\varphi(T)) = \mathbb{E} \sup_{t \in T} \sum_{i=1}^{\infty} \varphi_i(t_i)$.

Formulation in the Banach space

Let x_i, y_i, $i \geq 1$, be vectors in a Banach space $(B, \|\cdot\|)$. Suppose that for all $x^* \in B^*$ and $u \geq 0$,

$$\mathbb{P}(|\sum_{i\geq 1} x^*(x_i)\varepsilon_i| > u) \leq C\mathbb{P}(|\sum_{i\geq 1} x^*(y_i)\varepsilon_i| > C^{-1}u).$$

Equivalently, for any integer $p \geq 2$ and $x^* \in B^*$,

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Can we use the above to deduce the comparability of strong moments

$$\mathbb{E}\|\sum_{i\geq 1} x_i\varepsilon_i\| = \mathbb{E}\sup_{x^*\in B_1^*}\sum_{i\geq 1} x^*(x_i)\varepsilon_i \leq K\mathbb{E}\sup_{x^*\in B_1^*}\sum_{i\geq 1} x^*(y_i)\varepsilon_i = K\mathbb{E}\|\sum_{i\geq 1} y_i\varepsilon_i\|?$$

Results (W. Bednorz, R. M., 2019)

Recall the main assumption on the map φ i.e. that for any $s, t \in T$ and $p \geq 2$

$$\|B_{\varphi(t)} - B_{\varphi(s)}\|_p \leq \|B_t - B_s\|_p. \qquad (2.1)$$

Corollary

Suppose that $T = -T$, T is convex and $\mathrm{cl}(\mathrm{Lin}(T)) = \ell^2$. If φ is linear and satisfies (2.1) then $S_B(\varphi(T)) \leq KS_B(T)$, where K is a universal constant.

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$u \in \mathrm{Lin}(T)$ can be represented as $c \cdot t$, where $c \in \mathbb{R}$ and $t \in T$. By the linearity of φ,

$$\|B_{\varphi(u)}\|_p = |c|\|B_{\varphi(t)}\|_p \leq C|c|\|B_t\|_p = C\|B_u\|_p.$$

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Suppose that $T = -T$, T *is convex and* $\mathrm{cl}(\mathrm{Lin}(T)) = \ell^2$. *If* φ *is linear and satisfies (2.1) then* $S_B(\varphi(T)) \le K S_B(T)$, *where* K *is a universal constant.*

Corollary

Let x_i, y_i, $i \ge 1$, *be vectors in a Banach space* $(B, \|\cdot\|)$. *Let* $Q : B^* \to \ell^2$ *be defined by* $Q(x^*) = (x^*(y_i))_{i\ge1}$. *Suppose that* Q *is onto* ℓ^2. *If for any integer* $p \ge 2$ *and* $x^* \in B^*$, $\|\sum_{i\ge1} x^*(x_i)\varepsilon_i\|_p \le C\|\sum_{i\ge1} x^*(y_i)\varepsilon_i\|_p$, *then*

$$\mathbb{E}\|\sum_{i\ge1} x_i\varepsilon_i\| \le K\mathbb{E}\|\sum_{i\ge1} y_i\varepsilon_i\|.$$

Results (W. Bednorz, R. M., 2019)

Characterization of moments of B_t:

$$\|B_t\|_p \leq \sum_{i=1}^{p} |t_i^*| + \sqrt{p}\Big(\sum_{i>p} |t_i^*|^2\Big)^{\frac{1}{2}} \leq 4\|B_t\|_p,$$

where $(t_i^*)_{i\geq 1}$ is the rearrangement of $(t_i)_{i\geq 1}$ such that $|t_1^*| \geq |t_2^*| \geq \ldots$.
Equivalently,

$$\frac{1}{4}\inf_{t=t^1+t^2}(\|t^1\|_1 + \|G_{t^2}\|_p) \leq \|B_t\|_p \leq \inf_{t=t^1+t^2}(\|t^1\|_1 + \|G_{t^2}\|_p)$$

for $p \geq 2$.

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Main result (W. Bednorz, R. M., 2019)

Theorem

Suppose that for all $s, t \in T$ and all natural $p \geq 0$ we have

$$\inf_{|I^c| \leq Cp} \sum_{i \in I} |\varphi_i(t) - \varphi_i(s)|^2 \leq C^2 \inf_{|I^c| \leq p} \sum_{i \in I} |t_i - s_i|^2$$

for an absolute constant $C \geq 1$. Then $S_B(\varphi(T)) \leq K S_B(T)$, where K is a universal constant.

It improves the classic comparison result: take $\varphi(t)$ with short supports satisfying the assumption for $p = 0$.

Structure of $S_B(T)$

Recall that there exist T_1 and T_2 such that $T \subset T_1 + T_2$ and for some universal constant L we have

$$S_B(T) \geq L^{-1}\left(\sup_{t \in T_1} \|t\|_1 + S_G(T_2)\right).$$

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$S_G(T)$ is comparable with Talagrand's γ_2 functional which is related to the geometry of T:

$$\gamma_2(T) = \inf \sup_{t \in T} \sum_{n \geq 1} 2^{n/2} \Delta_2(A_n(t)).$$

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We provide a similar construction. Define

$$\gamma_X(T) = \inf \sup_{\pi\in C} \sum_{n\geq 1} \|X_{\pi_n} - X_{\pi_{n-1}}\|_{2^n},$$

where C (*chainings*) is a family of convergent sequences in T whose limits are dense in T and the infimum is taken over all chainings.

Structure of $S_B(T)$

- By means of $\gamma_X(T) = \inf \sup_{\pi \in C} \sum_{n \geq 1} \|X_{\pi_n} - X_{\pi_{n-1}}\|_{2^n}$ we can reformulate Bednorz-Latała bound in the following way. There exists a map $\tilde{\pi} : T \to \ell^2$ such that for $T_1 = \{t - \tilde{\pi}(t) : t \in T\}$ and $T_2 = \{\tilde{\pi}(t) : t \in T\}$ we have

$$\gamma_B(T_1 + T_2) \lesssim \gamma_B(T_1) + \gamma_B(T_2) \lesssim S_B(T) \leq 3\gamma_B(T).$$

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- Open question: can we formulate the lower bound in terms of $\gamma_B(T)$?
- If we could provide a counterexample to the hypothesis that $\|B_{\varphi(t)} - B_{\varphi(s)}\|_p \leq \|B_t - B_s\|_p$ for every $p \geq 2$ implies that $S_B(\varphi(T)) \leq L S_B(T)$, we would answer the question negatively.

Comparison on $T_1 + T_2$

As a consequence of the estimate: $S_B(T) \simeq \gamma_B(T_1 + T_2)$ we get

Corollary

Suppose that $\varphi : T \to \ell^2$ *can be extended to* $T_1 + T_2$ *in such a way that for any* $p \geq 2$,

$$\|B_{\varphi(t)} - B_{\varphi(s)}\|_p \leq \|B_t - B_s\|_p \ \textit{for all} \ s, t \in T_1 + T_2.$$

Then $S_B(\varphi(T)) \leq K S_B(T)$, *where* K *is a universal constant.*

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Then $S_B(\varphi(T)) \leq K S_B(T)$, *where* K *is a universal constant.*

$T_1 + T_2$ may be much larger than T!

Conjecture

Let $\varphi = (\varphi_i)_{i \geq 1} : T \to \ell^2$. If $\|B_{\varphi(t)} - B_{\varphi(s)}\|_p \leq \|B_t - B_s\|_p$, $p \geq 2$, $s, t \in T$, then

$$S_B(\varphi(T)) \lesssim K S_B(T).$$

Towards the counterexample

Comparison $S_B(\varphi(T)) \leq KS_B(T)$ holds in the following cases

- Gaussian part ("spread out part" $\sum_{i \in I} |\varphi_i(t) - \varphi_i(s)|^2$) of points from $\varphi(T)$ are dominated by the Gaussian part $\sum_{i \in I} |t_i - s_i|^2$ of points from T.

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- Supports $J(t) = \{i \geq 1 : |t_i| > 0\}$ of points in T are disjoint.

Corollary

Suppose that $\|B_{\varphi(t)} - B_{\varphi(s)}\|_p \leq \|B_t - B_s\|_p$, *for all* $p \geq 2$, $s, t \in T$ *and the supports* $J(t) = \{i \geq 1 : |t_i| > 0\}$ *are pairwise disjoint for all* $t \in T$. *Then* $S_B(\varphi(T)) \leq KS_B(T)$, *where* K *is a universal constant.*

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Corollary

Suppose that $\|B_{\varphi(t)} - B_{\varphi(s)}\|_p \le \|B_t - B_s\|_p$, for all $p \ge 2$, $s, t \in T$ and the supports $J(t) = \{i \ge 1 : |t_i| > 0\}$ are pairwise disjoint for all $t \in T$. Then $S_B(\varphi(T)) \le K S_B(T)$, where K is a universal constant.

The proof provides the intuitive split of the point t into the "peaky part" and the "spread out part", namely
$\pi(t) = t1_{J^2(t)}$ and $J^2(t) = \{i \in J(t) : |t_i| \le p(t)\}$
$t - \pi(t) = t1_{J^1(t)}$ where such that $J^1(t) \cup J^2(t) = J(t)$.

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$$\sup_{t^1 \in T_1} \|t^1\|_1 \leq LM \sup_{t \in T} \sum_{n=0}^{\infty} 2^n r^{-j_n(t)}$$

and

$$\gamma_G(T_2) \leq L\sqrt{M} \sup_{t \in T} \sum_{n=0}^{\infty} 2^n r^{-j_n(t)}.$$

Summary

There are still two questions to be answered:

1. Can the supremum of Bernoulli process be bounded in terms of the geometry of the set T only?
2. Is there a counterexample showing that the moment domination does not imply comparison of the suprema?

Summary

There are still two questions to be answered:

1. Can the supremum of Bernoulli process be bounded in terms of the geometry of the set T only?
2. Is there a counterexample showing that the moment domination does not imply comparison of the suprema?

Thank You!

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