Phase transition for the interchange and quantum Heisenberg models on the Hamming graph

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joint work with Radosław Adamczak and Piotr Miłoś

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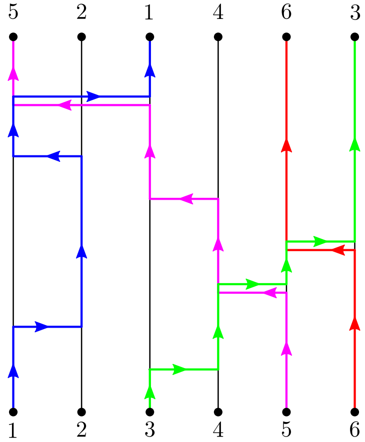
Main theme of the next 25 minutes

Understand the *cycle structure* of *random permutations*



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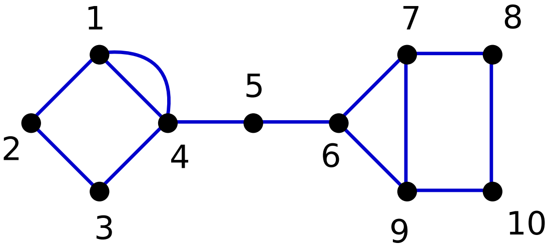

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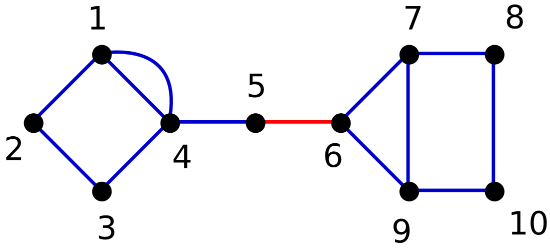

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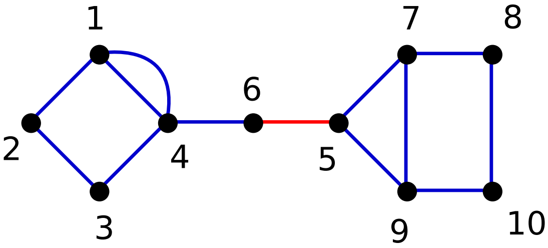

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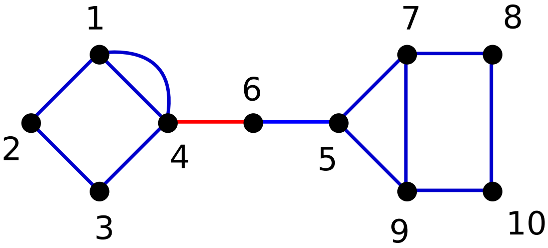

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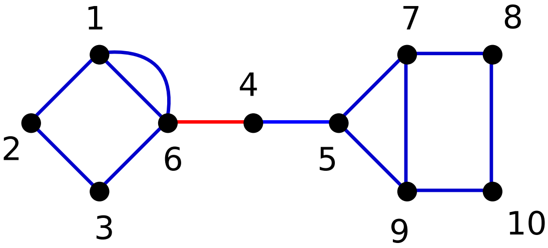

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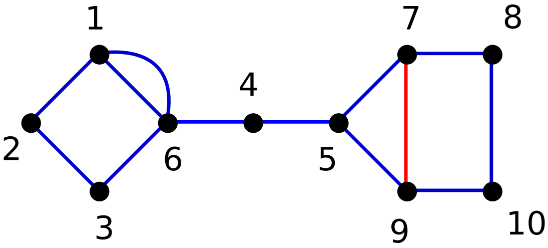

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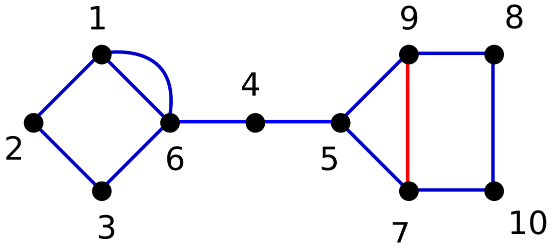

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- How does the picture depend on the geometry of G?

Random transpositions (card shuffling)

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- In particular: long cycles with positive probability after $\frac{1}{2} n \log n$ steps

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Total mixing time $\sim \frac{1}{2} n \log n \ldots$

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Poisson-Dirichlet distribution of macroscopic cycles

Furthermore, for $c > 1/2$ normalized cycle lengths converge to the Poisson-Dirichlet distribution

$$\left(\frac{X_1(\sigma_t)}{zn}, \frac{X_2(\sigma_t)}{zn}, \frac{X_3(\sigma_t)}{zn}, \dots\right) \Rightarrow \mathrm{PD}(1),$$

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Macroscopic cycles have the same distribution as in a uniformly random permutation from S_n.

Quantum Heisenberg ferromagnet

For a graph $G = (V, E)$ we consider a quantum spin system with Hamiltonian

$$H = -2 \sum_{(u,v)\in E} \left(\sigma_u^{(1)} \sigma_v^{(1)} + \sigma_u^{(2)} \sigma_v^{(2)} + \sigma_u^{(3)} \sigma_v^{(3)} \right)$$

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Physical observables can be expressed in terms of a *random permutation model*

Random loop representation

Fix $\theta > 0$ and reweight the probability of each transposition sequence:

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nonvanishing magnetization $\sim$ macroscopic cycles

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The Heisenberg ferromagnet on finite boxes of $\mathbb{Z}^d$, $d \geq 3$, exhibits a *phase transition* – nonvanishing spontaneous magnetization appears below critical temperature T_c.

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Probably very difficult!

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Our goal: analyze the process beyond the mean-field case (graphs with nontrivial geometry)

2-dimensional Hamming graph

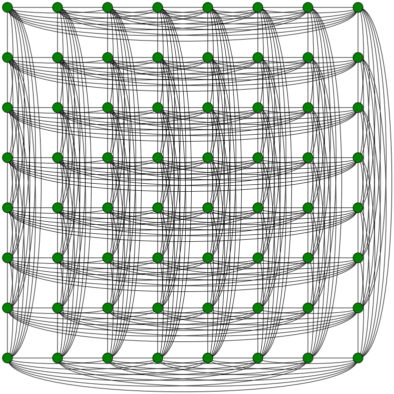

Figure: Hamming graph $H(2, n)$

$V = \{0, 1, \ldots, n-1\}^2$, $|V| = n^2$
$E = \{\text{complete graph in each row and column}\}$

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The first result of this kind for graphs with nontrivial geometry.

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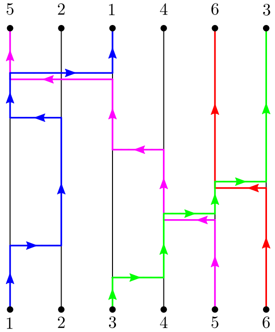
- Cyclic-time random walk



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Sharpness of the phase transition for $\theta \neq 1$?

The end

Thank you