HARDY INEQUALITY AND FRACTIONAL HARDY INEQUALITY FOR DUNKL LAPLACIAN

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Classical Hardy Inequality

- The classical Hardy inequality asserts that for any domain $\Omega \subset \mathbb{R}^N$, $(N \geq 3)$ with $0 \in \Omega$.

$$\int_\Omega |\nabla u|^2 \, dx \geq \lambda^* \int_\Omega \frac{|u|^2}{|x|^2} \, dx \quad \text{for all } u \in W_1^1(\Omega) \tag{1}$$

  where $\lambda^* := \frac{(N-2)^2}{4}$ is the optimal constant and is never achieved in $W_1^1(\Omega)$
- Usually the constant λ^* plays a crucial role in analysing the behaviour of heat equation with inverse square potential.
- The inequality (1) is strict for any $u \in W_1^1(\Omega)$.
- So we can think of improving (1). This was exploited by Brezis and V'azquez (Rev. Mat. Univ. Complut. Madrid, 1997) followed by Adimurthi, Chaudhuri, Ramaswamy(Proc.AMS, 2002) , Barbatis, Filippas, Tertikas(Indiana Univ. Math. J.,2003) and Pinchover, Tintarev (Indiana Univ. Math. J., 2005) to derive various improvements of (1) by imposing different conditions on Ω.

Improved Hardy Inequality

- For $\Omega = \mathbb{R}^N$ it has been shown by Devyver, Fraas, Pinchover (JFA,2014) and Fillipas, Tertikas (JFA 2002) that additional correction term cannot be added.
- Results of Brezis: For a bounded domain Ω in $\mathbb{R}^N$, $N \geq 3$ containing origin
- Result 1

$$\int_\Omega |\nabla u|^2 \, dx \geq \lambda^* \int_\Omega \frac{|u|^2}{|x|^2} \, dx + C_\Omega \int_\Omega |u|^2 \, dx. \tag{2}$$

- Result 2 For every $1 \leq q < 2$

$$\int_\Omega |\nabla u|^2 \, dx \geq \lambda^* \int_\Omega \frac{|u|^2}{|x|^2} \, dx + C_{q,\Omega} \left(\int_\Omega |\nabla u|^q \, dx \right)^{2/q}. \tag{3}$$

L^p Hardy Inequality

- For all $u \in W_p^1(\mathbb{R}^N)$ if $N > p$ and $u \in W_p^1(\mathbb{R}^N \setminus \{0\})$ if $N < p$

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx \geq \left(\frac{|N-p|}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} \, dx. \tag{4}$$

- One might ask whether inequality of the form

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \left(\frac{|N-p|}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} \, dx + C \left(\int_{\mathbb{R}^N} V(|x|)|u|^q \, dx\right)^{p/q} \tag{5}$$

  holds for some non trivial $V \geq 0$ and some $q > 0$.
- (Frank and Seiringer) For $p \geq 2$ and $N \neq p$ and with $w(x) = |x|^{-(N-p)/p}$ and $v(x) = |x|^{(N-p)/p} u(x)$

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \left(\frac{|N-p|}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} \, dx + c_p \int_{\mathbb{R}^N} \frac{|\nabla v|^p}{|x|^{N-p}} \, dx \tag{6}$$

Idea of Frank and Seiringer

- Case p=2: Write $u = vw$, $E[u] = \int_{\mathbb{R}^N} |\nabla u|^2$.
- integration by parts $\to$ $E[u] = \int_{\mathbb{R}^N} |\nabla v|^2 w^2 - \int_{\mathbb{R}^N} v^2 w \triangle w$.
- w is a positive weak solution of $-\triangle w = Vw$ for some function V.
- choose $w(x) = |x|^{-\frac{N-2}{2}}$, $V(x) = -\left(\frac{N-2}{2}\right)^2 |x|^{-2}$.
- Hardy inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} \, dx. \tag{7}$$

- The constant $\left(\frac{N-2}{2}\right)^2$ is sharp.

Idea of Frank and Seiringer $p \neq 2$

- $E[u] = \int_{\mathbb{R}^N} |\nabla u|^p, \; u = vw.$
- $1 \leq p \leq 2$, $|a+b|^p \geq |a|^p + p|a|^{p-2}a.b, \; a, b \in \mathbb{R}^N.$
- $p \geq 2$, $|a+b|^p \geq |a|^p + p|a|^{p-2}a.b + c_p|b|^p$
  $c_p = \min_{0<\tau<1/2}((1-\tau)^p - \tau^p + p\tau^{p-1})$
- $1 \leq p \leq 2$, integration by parts
  $E[u] \geq \int_{\mathbb{R}^N} |v|^p|\nabla w|^p - \int_{\mathbb{R}^N} w\mathrm{div}(|\nabla w|^{p-2}\nabla w)|v|^p$
- $p \geq 2$ integration by parts
  $E[u] \geq \int_{\mathbb{R}^N} |v|^p|\nabla w|^p - \int_{\mathbb{R}^N} w\mathrm{div}(|\nabla w|^{p-2}\nabla w)|v|^p + c_p \int_{\mathbb{R}^N} w^p|\nabla v|^p$
- Find positive weak solution w for $-\mathrm{div}(|\nabla w|^{p-2}\nabla w) = Vw^{p-1}$.
- $w(x) = |x|^{-\frac{N-p}{p}}, \; V(x) = \left(\frac{|N-p|}{p}\right)^p |x|^{-p}.$

Fractional Hardy Inequality

- Look back at the Hardy inequality:

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx$$

  for all $u \in W_1^1(\mathbb{R}^N)$.
- Rewrite it as

$$\int_{\mathbb{R}^N} -\triangle u(x)\overline{u(x)} \, dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx. \tag{8}$$

  Here $\triangle = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i}$.
- For any $0 < s < 1$ we can define $(-\triangle)^s$ by $\triangle^s u(x) = -\mathcal{F}^{-1}(|\cdot|^{2s}\hat{u})(x)$

Fractional Hardy Inequality

- $(-\triangle)^s u(x) = C(N,s) \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy.$
- $\langle (-\triangle)^s u, u \rangle = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dy$
- For $0 < s < 1$ and $s < \frac{N}{2}$

$$\|(-\triangle)^{s/2} u\|_2^2 = \langle (-\triangle)^s u, u \rangle \geq E_{N,s} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \, dx \tag{9}$$

$E_{N,s} = 4^s \frac{\Gamma(\frac{N+2s}{4})^2}{\Gamma(\frac{N-2s}{4})^2} \to \frac{(N-2)^2}{4}$ as $s \to 1$.

- In view of this

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \geq E_{N,s} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx.$$

Fractional Sobolev Space

- $p \neq 2$ one cannot have the equivalence of $\|(-\Delta^{s/2})u\|_p^p$ and $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy$
- $\Delta_p := div(|\nabla u|^{p-2}\nabla u)$
- fractional power of p-Laplacian

$$(-\Delta)_p^s u(x) := \lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy.$$

- We are interested in

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy \geq C'(N,s,p)\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}dx$$

- To find a Hardy inequality for the functional $E[u]$

$$E[u] := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^p k(x,y) dx dy$$

- The Euler-Lagrange equation of the functional $E[u]$ is

$$w(x)^{-p+1} 2 \int_{\mathbb{R}^N} |w(x) - w(y)|^{p-2}(w(x) - w(y)) k(x,y) dy = V(x).$$

- Look for family $k_\epsilon(x,y)$ symmetric $0 \leq k_\epsilon(x,y) \leq k(x,y)$, $\lim_{\epsilon \to 0} k_\epsilon(x,y) = k(x,y)$.
-

$$w(x)^{-p+1} 2 \int_{\mathbb{R}^N} |w(x) - w(y)|^{p-2}(w(x) - w(y)) k_\epsilon(x,y) dy = V_\epsilon(x) \tag{10}$$

- Assumption $V_\epsilon \to V$ weakly in $L^1_{loc}(\mathbb{R}^N)$.
- Multiply both sides of (10) by $|v(x)|^p w(x)^p$.
- Use the symmetry of the kernel $k_\epsilon(x,y)$ and take $\epsilon \to 0$.
- Use the simple inequality : $0 \leq t \leq 1$ $a \in \mathbb{C}$, then for $p \geq 1$ $|a - t|^p \geq (1 - t^{p-1})(|a|^p - t)$ and for $p \geq 2$ $|a - t|^p \geq (1 - t^{p-1})(|a|^p - t) + c_p t^{p/2} |a - 1|^p$, with $0 < c_p \leq 1$.

Sharp Hardy inequality with remainder

- $\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \Phi_u(x,y) + \int\limits_{\mathbb{R}^N} V|u|^p = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} |u(x)-u(y)|^p k(x,y)$, $u = vw$.
- For $p \neq Ns$ let $\alpha = \frac{N-ps}{p}$, $k(x,y) = |x-y|^{-N-ps}$, $w(x) = |x|^{-\alpha}$ $V(x) = C_{N,s,p}|x|^{-ps}$. Euler-Lagrange equation is satisfied.
- The kernel $k(x,y) = |x-y|^{-(N+ps)}$ is symmetric and translation of the function $|x|^{-(N+ps)}$. Write $k_\epsilon(x,y) = k(x,y)$ if $||x|-|y|| > \epsilon$ otherwise zero.
- Write $G_{s,p}(u) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dxdy - C_{N,s,p}\int\limits_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}}\, dx$
- Frank and Seiringer : For $p \geq 2$, there exits a positive constant $\Lambda_{N,s,p}$ such that for all $u \in C_c^\infty(\mathbb{R}^N)$

$$G_{s,p}(u) \geq c_p \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}\, w^{p/2}(x) w^{p/2}(y) dxdy, \qquad (11)$$

  where $w(x) = |x|^{-\frac{|N-ps|}{2}}$ and $v(x) = \frac{u(x)}{w(x)}$.
- For $p = 2$ equality holds with $c_p = 1$.

Preliminaries of Dunkl theory

For $\alpha \in \mathbb{R}^N \setminus \{0\}$, we denote σ_α as the reflection in the hyper plane $\langle\alpha\rangle^\perp$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle\alpha, x\rangle}{|\alpha|^2}\alpha,$$

where $|\alpha| := \sqrt{\langle\alpha, \alpha\rangle}$.

Definition

Let $R \subset \mathbb{R}^N \setminus \{0\}$ be a finite set. Then R is called a root system, if
(1) $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in R$
(2) $\sigma_\alpha(R) = R$ for all $\alpha \in R$.

Preliminaries of Dunkl theory

- The group $G = G(R)$ which is generated by reflections $\{\sigma_\alpha : \alpha \in R\}$ is called reflection group (or Coxeter-group) associated with R.
- A $G-$ invariant function from the root system R to $\mathbb{R}_+$ is called a multiplicity function.
- $R = R_+ \sqcup (-R_+)$ where R_+ and $-R_+$ is separated by a hyperplane through the origin. Such a set R_+ is called positive subsystem.

Preliminaries of Dunkl theory

- Let ∂_j denotes the partial derivative corresponding to e_j, and R is a fixed root system.

Definition

Let k be a multiplicative function. Then for $e_j \in \mathbb{R}^N$, the Dunkl operator $T_j := T_{e_j}(k)$ is defined (for $f \in C^1(\mathbb{R}^N)$) by

$$T_j f(x) := \partial_j f(x) + \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, e_j \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$

- Associated with the reflection group and the function k, the weight function h_k is defined by $h_k(x) = \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{k_\nu}, \quad x \in \mathbb{R}^N$.
- This is a positive homogeneous function of degree $\gamma_k := \sum_{\nu \in R_+} k_\nu = \sum_{\nu \in R_+} k(\nu)$ and is invariant under the reflection group G. Let us denote $d_k := N + 2\gamma_k$; $d\mu_k(x) := h_k^2(x)dx$.

Preliminaries of Dunkl theory

- If $f, g \in C^1(\mathbb{R}^N)$ and at least one of them is G-invariant then $T_j(fg) = T_j f.g + f.T_j g$.
- The Dunkl kernel $E_k(., y)$ is defined as the unique solution of the system,

$$T_j f = y_j f, \quad f(0) = 1.$$

- $E_k(x, y)$ is the analogous function of $e^{\langle x,y\rangle}$ as $e^{\langle .,y\rangle}$ is the solution of $\partial_j f = y_j f$ with $f(0) = 1$.
- For $u \in L^1(\mathbb{R}^N, d\mu_k(x))$, its Dunkl Fourier transform is defined by

$$\mathcal{F}_k u(\xi) = c_h^{-1} \int\limits_{\mathbb{R}^N} u(x) E_k(-i\xi, x) d\mu_k(x)x,$$

where $c_h := \left(\int_{\mathbb{R}^N} e^{-\|x\|^2/2} d\mu_k(x) \right)^{-1}$.

- $\mathcal{F}_k(\tau_y^k f)(\xi) = E_k(iy, \xi)\mathcal{F}_k f(\xi)$
- $\nabla_k = (T_1, T_2, \cdots, T_N)$, $\Delta_k = \sum\limits_{j=1}^{N} T_j^2$

Technical difficulties

- Dunkl weighted measure is not invariant under translation.
- Product rule works only when one of the functions is G-invariant.
- As result of this $\nabla_k u^2$ need not be equal to $2\nabla_k u$.
- For $p \neq 2$ boundedness of the operator τ_y^k on $L^p(\mathbb{R}^N)$ is not known.

Observation:

- Radial functions, in general $G-$invariant functions, are more easy to deal with Dunkl setting. In the case of classical Hardy inequality, we found that a radial function $w(x) := |x|^{-\lambda_k}$ will do the work.
- τ_y^k is bounded on $L^p_{rad}(\mathbb{R}^N)$, $1 \leq p < \infty$.
- $f(x) = f_0(|x|)$; $\tau_y^k f(x) = \int_{\mathbb{R}^N} f_0(\sqrt{|x|^2 + |y|^2 - 2\langle y, \eta\rangle}) d\mu_k^x(\eta)$

Theorem

Let $1 \leq p < \infty$. Let u be a real valued $G-$ invariant function. If $u \in C_0^\infty(\mathbb{R}^N)$ if $d_k > p$ and $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $d_k < p$ then the following inequality holds:

$$\int_{\mathbb{R}^N} |\nabla_k u(x)|^p d\mu_k(x) \geq \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} d\mu_k(x).$$

The constant $\left|\frac{d_k-p}{p}\right|^p$ given in the inequality is optimal. For $p \geq 2$

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k(x) - \left|\frac{d_k - p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} d\mu_k(x) \geq c_p \int_{\mathbb{R}^N} \frac{|\nabla_k v|^p}{|x|^{d_k - p}} d\mu_k(x),$$

where c_p is given by

$$c_p := \min_{0<\tau<1/2} \left((1-\tau)^p - \tau^p + p\tau^{p-1}\right).$$

When $p = 2$ the equality holds and with $c_2 = 1$.

Weighted Fractional hardy type inequality in Dunkl setting

- Observation $\frac{1}{|x-y|^\alpha}$ is the translation of $\frac{1}{|x|^\alpha}$ and it is symmetric.
- $|x-y|^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int\limits_0^\infty s^{\alpha/2-1} e^{-s|x-y|^2}\ ds$
- $|x-y|^{-\alpha}$ will be replaced by

$$\Phi_\alpha(x,y) = \frac{1}{\Gamma(\alpha)} \int\limits_0^\infty s^{\alpha/2-1} \tau_k^x e^{-s|y|^2}\ ds$$

- It can be shown $\int\limits_{\mathbb{R}^N} \triangle_k^s u(x) v(x)\ d\mu_k(x) =$ $\int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} (u(x)-u(y))(v(x)-v(y))\Phi_{d_k+2s}(x,y)\ dx\ dy$
- $\triangle_p^s$ will be replaced by $(\triangle_k)_p^s$ by changing $|x-y|^{-(N+ps)}$ to $\Phi_{d_k+ps}(x,y)$

Theorem

Let $d_k \geq 1$ *and* $0 < s < 1$*. If* $u \in \dot{W}^s_{k,p}(\mathbb{R}^N)$ *when* $p < d_k/s$ *or* $u \in \dot{W}^s_{k,p}(\mathbb{R}^{\mathbb{N}} \setminus \{0\})$ *when* $p > d_k/s$*, the following inequality holds;*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{d_k+ps}(x,y)\,d\mu_k(x)\,d\mu_k(y) \geq C_{d_k,s,p}\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}}\,d\mu_k(x$$

where

$$C_{d_k,s,p} := 2\int_0^1 r^{ps-1}|1-r^{(d_k-ps)/p}|^p \Phi_{N,s,p}(r)\,dr,$$

with

$$\Phi_{N,s,p}(r) := \frac{\Gamma(\frac{d_k}{2})}{\sqrt{\pi}\,\Gamma(\frac{d_k-1}{2})}\int_0^{\pi} \frac{sin^{d_k-2}\theta}{(1-2r\ cos\ \theta + r^2)^{\frac{d_k+ps}{2}}}\,d\theta,\ \ N \geq 2,$$

$$\Phi_{1,s,p}(r) := \Big(\tau^k_r\big(|.|^{d_k+ps}\big) + \tau^k_{-r}\big(|.|^{d_k+ps}\big)\Big)(1),\ \ N = 1.$$

Theorem (continued)

The constant $C_{d_k,s,p}$ is sharp. If $p=1$, equality holds iff u is proportional to a symmetric decreasing function. If $p>1$, the inequality is strict for any function $0 \not\equiv u \in \dot{W}^s_{k,p}(\mathbb{R}^N)$ or $\dot{W}^s_{k,p}(\mathbb{R}^N \setminus \{0\})$, respectively. Further for $p \geq 2$ the following inequality holds.

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p \Phi_{d_k+ps}(x,y)\,d\mu_k(x)\,d\mu_k(y)$$
$$\geq C_{d_k,s,p}\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}}\,d\mu_k(x)$$
$$+c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(x)-v(y)|^p \Phi_{d_k+ps}(x,y)\frac{d\mu_k(x)}{|x|^{(d_k-ps)/2}}\frac{d\mu_k(y)}{|y|^{(d_k-ps)/2}},$$

where $v := |x|^{(d_k-ps)/p} u$ and c_p is given in (12). $c_2 = 1$ and the equality holds in $p=2$ case.

References

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THANK YOU

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