On the affine recursion in dimension ≥ 2 in the "centered" case

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Probability and Analysis

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Plan

1. Introduction
   - The affine recursion
   - Contracting case (in dimension 1)
   - Centered case (in dimension 1)
   - Extension to iterated function systems

2. The affine recursion in dim ≥ 2

3. Elements of proof

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The affine recursion

We consider the Markov chain $(X_n)_{n\geq 0}$ on $(\mathbb{R}^+)^d$ defined inductively by: for any $n \geq 1$,

$$X_{n+1} = A_{n+1}X_n + B_{n+1}$$

where the (A_n, B_n) are i.i.d. random variables
• the A_n are $d \times d$ matrices with positive entries;
• the B_n are vectors in $(\mathbb{R}^+)^d$.
For any $n \geq 1$

$$X_n = A_{n,1}X_0 + B_{n,1}$$

with $A_{n,1} = A_n \cdots A_1$ and $B_{n,1} = \sum_{k=1}^{n} A_n \cdots A_{k+1} B_k$.
When $X_0 = x$ for some fixed $x \in \mathbb{R}^d$, we set $X_n = X_n^x$.

Contracting case (in dimension 1)

Assume that $\mathbb{E}(\ln a_1|) < +\infty, \mathbb{E}(\ln^+ |b_1|) < +\infty$ *and*

$$\mathbb{E}(\ln a_1) < 0.$$

Then, there exists an unique invariant probability measure ν *on* $\mathbb{R}$ *for* $(X_n)_{n\geq 1}$*.*

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Sketch of the proof. Set $g_i(x) := a_i x + b_i, i \geq 1$. It holds $X_n^x = g_n \circ \ldots \circ g_1(x)$.
Now, set $Z_n^x = g_1 \circ \ldots \circ g_n(x) = a_1 \ldots a_n x + (b_1 + a_1 b_2 + a_1 a_2 b_3 + \ldots + a_1 \ldots a_{n-1} b_n.)$

$$a_1 \ldots a_n \to 0 \quad \text{and} \quad (a_1 \ldots a_{n-1}|b_n|)^{1/n} \to \exp(\ln a_1) < 1 \quad \text{a.s.}$$

Then $Z_n^x \longrightarrow Z_\infty$ a.s. and $\mathcal{L}(X_n^x) = \mathcal{L}(Z_n^x) \longrightarrow \nu := \mathcal{L}(Z_\infty)$. □

Centered case (in dimension 1)

(Babillot M., Bougerol Ph. Elie L.) (1998) Brofferio (2003)

Assume that

- *(Non degeneracy assumption)* $\forall x \in \mathbb{R} \quad \mathbb{P}(a_1 x + b_1 = x) < 1$ *and* $\mathbb{P}(a_1 \neq 1) > 0$.
- $\mathbb{E}((\ln a_1)^2)| < +\infty, \mathbb{E}(\ln^+ |b_1|^{2+\epsilon}) < +\infty$
- $\mathbb{E}(\ln a_1) = 0$.

Then, there exists a unique invariant (infinite) radon measure λ *on* $\mathbb{R}$ *for* $(X_n)_{n\geq 1}$.

Sketch of Babillot, Bougerol & Elie's proof

- **Conservativity** (⇒ existence of λ): for some (large enough) compact set $K \subset \mathbb{R}$,

$$\sum_{n \geq 0} \mathbb{P}(S_n \in K) = \infty.$$

Based on the fact that $\mathbb{P}(\tau > n) \sim \frac{c}{\sqrt{n}}$ as $n \to +\infty$, where $\tau := \inf\{n \geq 1 : a_1 \ldots a_n < 1\}$.

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- **Local contractivity** (⇒ unicity of λ): for any $x, y \in \mathbb{R}$ and any compact set $K \subset \mathbb{R}$,

$$|X_n^x - X_n^y| 1_K(X_n^x) = a_1 \dots a_n |x - y| 1_K(X_n^x) \quad \xrightarrow{a.s.} \quad 0.$$

Based on the fact that $\mathrm{Aff}(\mathbb{R}) = \{x \mapsto ax + b\}$ is non uni-modular, so the random walk $(g_1 \circ \dots \circ g_n)_{n \geq 1}$ is transient + Renewal theorem. □

Iterated function systems

Let $(f_n)_{n\geq 1}$ be i.i.d. random variables with values in $\mathcal{C}(\mathbb{R}^d)$ and set, for any $x \in \mathbb{R}^d$,

$$X_n^x := f_n \circ \ldots \circ f_1(x).$$

Theorem (M. Benda (cf Woess & P. (2011)) *Assume conservativity and local contractivity. Then, there exists on $\mathbb{R}^d$ a unique (finite or infinite) invariant Radon measure for $(X_n^x)_{n\geq 0}$.*

Applications. Affine recursion in dimension 1, random walk on $\mathbb{R}^+$ with elastic or non elastic reflexion at 0, a mix of these two recursions...

The affine recursion in dim ≥ 2: notations

Fix $\Delta > 0$. Let

- S_+ be the set of $d \times d$ matrices $A = (A(i,j))_{1 \leq i,j \leq d} \in S$ with non-negative entries;
- S_Δ be the subset of matrices $A \in S_+$ s.t. for any $1 \leq i, j, k, l \leq d$,

$$\frac{1}{\Delta} A(k,l) \leq A(i,j) \leq \Delta A(k,l);$$

- $\mathcal{C} = (\mathbb{R}^+)^d$;
- $\mathbb{X} := \{x = (x_i)_{1 \leq i \leq d} \in \mathcal{C} : \ x_1 + \ldots + x_d = 1\}$.

Let $g_n = (A_n, B_n)$ be i.i.d. with values in $S_\Delta \times (\mathbb{R}^+)^d$.
Assume $\mathbb{E}(|\ln|A_1||) < +\infty$. The **Lyapunov exponent** of the matrices $(A_n)_{n\geq 1}$ is

$$\gamma := \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}(\ln|A_n \dots A_1|).$$

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Lemma (Furstenberg-Kesten) *For any matrix A in the semi-group T_{S_Δ} generated by S_Δ and any* $1 \leq i, j, k, l \leq d$,

$$\frac{1}{\Delta^2} A(k,l) \;\leq\; A(i,j) \;\leq\; \Delta^2 A(k,l).$$

In particular, there exists a constant $c = c(\Delta) > 1$ such that for any $A, B \in T_{S_\Delta}$ and any $x \in \mathcal{C}, |x| = 1$,

1. $\frac{1}{c}|A| \leq |Ax| \leq |A|$,
2. $\frac{1}{c}|A|\,|B| \;\leq\; |AB| \;\leq\; |A||B|$.

Main Theorem

(Brofferio, P., Pham) Assume that

A1- There exists $\delta_1 > 0$ s.t. $\mathbb{E}((\ln|A_1|)^{2+\delta_1}) < +\infty$.
A2- (Irreducibility) There exists no affine subspaces $\mathcal{A}$ of $\mathbb{R}^d$ such that $\mathcal{A} \cap (\mathbb{R}^+)^d$ is non-empty, bounded and invariant under the action of all elements of the support of the distribution of A_1.
A3- The upper Lyapunov exponent γ of the A_n equals 0.
A4- There exists $\delta_2 > 0$ such that $\mathbb{P}\{|A_1 x| \geq (1+\delta_2)|x| \quad \forall x \in \mathcal{C}\} > 0$.

B- The random variables B_k are $(\mathbb{R}^+)^d$-valued and there exists $\delta_3 > 0$ such that

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Then, there exists on $\mathcal{C}$ a unique invariant (infinite) Radon measure λ for $(X_n^x)_{n\geq 0}$.

Proof. Step 1. Conservativity

For $x \in \mathcal{C}$ and $a > 1$, set

$$\tau_{x,a} := \min\{n \geq 1 : \; a|\overbrace{A_n \cdots A_1}^{A_{n,1}} x| \leq 1\}$$

and

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Fix $a > 1$ and set $\tau^{(1)} = \tau_a$ and $\tau^{(k+1)} = \min\{n \geq 1 : a|A_n \cdots A_{\tau^{(k)}}| \leq 1\}$.
The product of matrices

$$A_{\tau^{(k)}} \ldots A_1$$

may be decomposed as a product of k i.i.d. random matrices

$$\overbrace{\left(A_{\tau^{(k)}} \ldots A_{\tau^{(k)-1}+1}\right)}^{\widetilde{A}_k} \overbrace{\left(A_{\tau^{(k)-1}+1} \ldots A_{\tau^{(k)-2}+1}\right)}^{\widetilde{A}_{k-1}} \ldots\ldots\ldots \overbrace{\left(A_{\tau} \ldots A_1\right)}^{\widetilde{A}_1}.$$

Proof. Step 1.

$$X_{\tau^{(k)}} = A_{\tau^{(k)}} \dots A_1 x + \sum_{j=1}^{\tau^{(k)}-1} A_{\tau^{(k)}} \dots A_{j+1} B_j$$

$$= A_{\tau^{(k)}} \dots A_1 x + \sum_{l=1}^{k} A_{\tau^{(k)}} \cdots A_{\tau^{(\ell)}+1} \left(\sum_{j=\tau^{(\ell-1)}+1}^{\tau^{(\ell)}} A_{\tau^{(\ell)}} \dots A_{j+1} B_j \right)$$

$$= \widetilde{A_k} \dots \widetilde{A_1} x + \sum_{l=1}^{k} \widetilde{A_k} \dots \widetilde{A_{l+1}} \left(\sum_{j=\tau^{(\ell-1)}+1}^{\tau^{(\ell)}} A_{\tau^{(\ell)}} \dots A_{j+1} B_j \right)$$

$$= \widetilde{A_k} \dots \widetilde{A_1} x + \sum_{l=1}^{k} \widetilde{A_k} \dots \widetilde{A_{l+1}} \widetilde{B}_\ell$$

where the $(\widetilde{A}_k, \widetilde{B}_k)$ are i.i.d. and satisfy $|\widetilde{A}_k| \leq \frac{1}{a} < 1$.

Question: $\mathbb{E}\left(\ln^+(|\widetilde{B_1}|)\right) < +\infty$?

Fluctuations of the norm of matrices

For $x \in \mathcal{C}$ and $a \geq 1$, set
$\tau_{x,a} := \min\{n \geq 1 : a|A_n \cdots A_1 x| \leq 1\}$ and $\tau_a := \min\{n \geq 1 : a|A_n \cdots A_1| \leq 1\}$.

Theorem (Grama I., Le page E. & P (2017), Pham. T.D.C, 2018)
Assume hypotheses **A**.
Then, there exists an harmonic function on $\mathbb{X} \times \mathbb{R}^+$ for the chain $(A_{n,1} \cdot x, \ln |A_{n,1} x|))$ restricted to $\mathbb{X} \times \mathbb{R}^+$ such that

$$\frac{1}{c} \vee (\ln a - \text{cste}) \leq V(x,a) \leq c(1 + \ln a) \quad \text{and} \quad V(x,a) \sim \ln a \quad \text{as } a \to +\infty,$$

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For any $x \in \mathbb{X}$ and $a \geq 0$,

$$\mathbb{P}(\tau_{x,a} > n) \sim \frac{2V(x,a)}{\sigma\sqrt{2\pi n}} \text{ as } n \to +\infty.$$

Moreover, there exists a constant $c > 0$ such that for any $x \in \mathbb{X}$, $a \geq 0$ and $n \geq 1$,

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Consequence:

- $\mathbb{P}(\tau_a > n) \leq c' \frac{(1+\ln a)}{\sqrt{n}}$ for some $c' > 0$; in particular $\mathbb{E}(\tau_a^{\frac{1}{2}-\epsilon}) < +\infty$;
- $\mathbb{E}\left(\ln^+(|\widetilde{B_1}|)\right) < +\infty$ (by Elie's argument).

Ideas of the proof of Pham's result: the D. Denisov & V. Wachtel's strategy

Step 1. $\ln a|A_{n,1}x|$ may be decomposed as

$$S_n(x,a) = \ln a + \rho(A_1,x) + \rho(A_2,\xi_1) + \ldots + \rho(A_n,\xi_{n-1}) \quad \text{with} \quad \xi_k = \frac{A_{k,1}x}{|A_{k,1}x|} \in \mathbb{X}$$

The sequence $(\xi_n, S_n(x,a)))_{n\geq 0}$ is a semi-Markovian random walk on $\mathbb{X}\times\mathbb{R}$.

Step 2. Description of the spectral properties of the transition operator of $(\xi_n)_{n\geq 0}$ (spectral gap property via Doeblin-Fortet theorem)

$$S_n(x,a) \longrightarrow M_n \qquad V(x,a) = \lim_{n\to+\infty} \mathbb{E}(S_n(x,a);\tau_{x,a}>n) = a - \mathbb{E}(M_{\tau_{x,a}})$$

Step 3. Approximation of the continuous time process

$$\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]} S_k(x,a)\right)_{t\in[0,1]}$$

by a Brownian motion, via **a weak invariant principle for dependent random variables** (Sakhanenko's theorem in the i.i.d. case)

Proof. Step 2. Local contractivity

If $\mathbb{P}(B_1 \neq 0) > 0$ and $\sum_{n=0}^{+\infty} 1_{[|A_{n,1}| \leq 1]} = +\infty$ $\mathbb{P}$-a.s., then, $\mathbb{P}$-a.s., for any $x, y \in \mathbb{R}_+^d$ and any $K > 0$,

$$\lim_{n \to +\infty} |X_n^x - X_n^y| 1_{[|X_n^x| \leq K]} = 0.$$

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Proof. • $|X_n^x - X_n^y| 1_{[|X_n^x| \leq K]} \leq |A_{n,1}||x - y| 1_{[|X_n^x| \leq K]} \leq \frac{K}{\frac{|X_n^x|}{|A_{n,1}|}} |x - y|.$

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• $\dfrac{|X_n^x|}{|A_{n,1}|} \geq \dfrac{|B_{n,1}|}{|A_{n,1}|} = \sum_{k=1}^{n} \dfrac{|A_{n,k+1} B_k|}{|A_{n,1}|} \geq \dfrac{1}{c} \sum_{k=1}^{n} \dfrac{|A_{n,k+1}||B_k|}{|A_{n,1}|} \geq \dfrac{1}{c} \sum_{k=1}^{n} \dfrac{|B_k|}{|A_{k,1}|}$

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with

$$\sum_{k=1}^{n} \frac{|B_k|}{|A_{k,1}|} \geq \sum_{k=1}^{n} \frac{|B_k|}{|A_k|} \frac{1}{|A_{k-1,1}|} \geq \delta \sum_{k=1}^{n} \varepsilon_k \eta_k \xrightarrow{\mathbb{P}-\text{a.s.}} +\infty.$$

(where we set $\varepsilon_k = 1_{[|B_k / A_{k,1}| \geq \delta]}$, $\eta_k = 1_{[|A_{k-1,1}| \leq 1]}$).

□

Questions

- Properties of the invariant measure λ.
- Case when $A_i \in S^+$ and $B_i \in \mathbb{R}^d$.
- Case when $A_i \in GL(d, \mathbb{R})$ or subset of $M(d, \mathbb{R})$
- Other iterated functions systems