“The A_∞ class of weights and some of its extensions”

Carlos Pérez

University of the Basque Country and BCAM

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Banach Center for Mathematics

Będlewo, May 22, 2019

Collaborators

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- new results joint with

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Javier Canto

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S. Ombrosi, E. Rela and I. Rivera-Rios

El teorema de Muckenhoupt

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Thm (B. Muckenhoupt (≈ 1971))

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$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w \, dx \right) \left(\frac{1}{|Q|} \int_Q w^{\frac{-1}{p-1}} \, dx \right)^{p-1} < \infty$$

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This result was greatly improved.

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Corollary Let $p \in (1, \infty)$ and let $w \in A_p$. Then

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- Key observation: $(M\mu)^{-\lambda} \in A_\infty$

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• (work with D. Cruz-Uribe, JM Martell).

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- Hence: $A_\infty \subset C_p$

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$$w(E) \le C \left(\frac{|E|}{|Q|}\right)^{\delta} \int_{\mathbb{R}^n} (M\chi_Q(x))^p w(x)\,dx \qquad E \subset Q$$

• Compare with the A_∞ condition: $w(E) \le c \left(\frac{|E|}{|Q|}\right)^{\delta} w(Q) \qquad E \subset Q$

• Hence: $A_\infty \subset C_p$

• Open problem, **Is the C_p condition sufficient?**

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