Sharp Hardy's inequality for Laguerre and Hermite expansions

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Classical Hardy's inequality

Theorem (Hardy, Littlewood, 1927)

For $f \in \mathrm{Re} H^1$ it holds

$$\sum_{k \in \mathbb{Z}} \frac{|\hat{f}(k)|}{|k|+1} \lesssim \|f\|_{\mathrm{Re} H^1},$$

where $\hat{f}(k)$ is k-th Fourier coefficient of f.

Here $\mathrm{Re} H^1$ denotes the real Hardy space constituted by the boundary values of the real parts of functions in the Hardy space $H^1(\mathbb{D})$, where $\mathbb{D}$ is the unit disk on the plane.

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Kanjin's results

Hardy's inequality for Hermite and (standard) Laguerre expansions, (Kanjin '97)

$$\sum_{k\in\mathbb{N}} \frac{|\langle f, h_k\rangle_{L^2(\mathbb{R})}|}{(k+1)^{29/36}} \lesssim \|f\|_{H^1(\mathbb{R})}, \qquad \sum_{k\in\mathbb{N}} \frac{|\langle f, \mathcal{L}_k^{\alpha}\rangle_{L^2(\mathbb{R}_+)}|}{k+1} \lesssim \|f\|_{H^1(\mathbb{R}_+)}.$$

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Other results: R. Balasubramanian, Y. Kanjin , Z. Li, R. Radha, M. Satake, K. Sato, Y. Shi, S. Thangavelu, Y. Yu.

Main aim

For a suitable orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}^d}$ in $L^2(X,\mu)$, we want to study inequality of the form

$$\sum_{n\in\mathbb{N}^d} \frac{|\langle f,\varphi_n\rangle|}{(n_1+\ldots+n_d+1)^E} \lesssim \|f\|_{H^1(X,\mu)}, \qquad f\in H^1(X,\mu),$$

where $n=(n_1,\ldots,n_d)$, the symbol $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(X,\mu)$, and $H^1(X,\mu)$ is an appropriate Hardy space.

The papers

- P. P., *Hardy's type inequality for Laguerre expansions of Hermite type*, J. Fourier Anal. Appl. (2018).
- P. P., *Sharp Hardy's type inequality for Laguerre expansions*, preprint (2018), arXiv:1810.08138.
- P. P., *On Hardy's inequality for Hermite expansions*, preprint (2019), arXiv:1901.05663.

Considered setting

Let (X, μ) be a measure metric space such that X is an open convex subset of $\mathbb{R}^d$, the measure μ is doubling, and the space is equipped with the Euclidean metric.

Note that this implies that (X, μ) is a space of homogeneous type in the sense of Coifman and Weiss.

We also impose the following assumption (lower Ahlfors' condition): there exists N such that

$$\mu(B(x, \rho)) \gtrsim \rho^N,$$

uniformly in $x \in X$ and $\rho \in (0, \operatorname{diam}(X))$, where $B(x, \rho) = \{y \in X\colon \sum_{i=1}^{d} (x_i - y_i)^2 < \rho^2\}$.

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Hardy space

The $(1, 2)$-atoms in the sense of Coifman and Weiss are measurable functions a supported in balls $B(x_0, \rho)$, such that

$$\int_B a(x)\, d\mu(x) = 0, \qquad \|a\|_{L^2(X,\mu)} \leq \mu(B)^{-1/2}.$$

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If $\mu(X) < \infty$, then, additionally, $a \equiv \mu(X)^{-1/2}$ is also considered as an $H^1(X,\mu)$-atom.

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The (atomic) Hardy space $H^1(X,\mu)$ is composed of functions $f \in L^1(X,\mu)$ admitting the atomic decomposition

$$f = \sum_{j=0}^{\infty} \lambda_j a_j,$$

where a_j's are $H^1(X,\mu)$-atoms and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$.

We stress that $H^1(X,\mu)$ is a Banach space with the norm

$$\|f\|_{H^1(X,\mu)} = \inf \sum_{j=0}^{\infty} |\lambda_j|,$$

where the infimum is taken over all atomic decompositions of f.

It is worth mentioning that for every $f \in H^1(X,\mu)$ there is

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Let $\{\varphi_n\}_{n\in\mathbb{N}^d}$ be an orthonormal basis in $L^2(X,\mu)$ such that $\varphi_n \in L^\infty(X,\mu)$, $n \in \mathbb{N}^d$.

We define the family of operators $\{R_r\}_{r\in(0,1)}$ via

$$R_r f = \sum_{n\in\mathbb{N}^d} r^{|n|}\langle f, \varphi_n\rangle \varphi_n, \qquad r \in (0,1), \qquad f \in L^2(X,\mu).$$

Notice that for every $r \in (0,1)$ the operator R_r is a contraction on $L^2(X,\mu)$.

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We assume that R_r are integral operators and the associated kernels are denoted by $R_r(x,y)$, namely

$$R_r f(x) = \int_X R_r(x,y) f(y)\, d\mu(y).$$

Moreover, we assume for some $\gamma > 0$ and a finite set Δ composed of positive numbers the condition

$$\|R_r(x,\cdot) - R_r(x',\cdot)\|_{L^2(X,\mu)} \lesssim \sum_{\delta\in\Delta} |x - x'|^{\delta} (1-r)^{-\frac{\gamma(N+2\delta)}{N+2}},$$

uniformly in $r \in (0,1)$, $x' \in X$, and almost every x such that $|x' - x| \leq 1/3$.

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Theorem (Hardy's inequality for orthogonal expansions)

Let (X,μ), $\{\varphi_n\}_{n\in\mathbb{N}^d}$, and $\{R_r\}_{r\in(0,1)}$ be as described. The inequality

$$\sum_{n\in\mathbb{N}^d}\frac{|\langle f,\varphi_n\rangle|}{(|n|+1)^E}\lesssim\|f\|_{H^1(X,\mu)},$$

holds uniformly in $f\in H^1(X,\mu)$, where

$$E=\frac{\gamma N}{N+2}+\frac{d}{2}.$$

Standard Laguerre expansions

The one-dimensional *standard Laguerre functions* $\{\mathcal{L}_k^\alpha\}_{k\in\mathbb{N}}$ of order $\alpha \geq 0$ are

$$\mathcal{L}_k^\alpha(x) = \left(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} L_k^\alpha(x) x^{\alpha/2} e^{-x/2}, \qquad x > 0.$$

In the multi-dimensional situation set

$$\mathcal{L}_n^\alpha(x) = \prod_{i=1}^{d} \mathcal{L}_{n_i}^{\alpha_i}(x_i), \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d.$$

These functions form an orthonormal basis in $L^2(\mathbb{R}_+^d, dx)$.

For Lebesgue measure we have $N = d$.

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The one-dimensional kernels associated with the operators R_r have the explicit representation

$$R_r^\alpha(x,y) = (1-r)^{-1} r^{-\alpha/2} \exp\Big(-\frac{1}{2}\frac{1+r}{1-r}(x+y)\Big) I_\alpha\Big(\frac{2r^{1/2}}{1-r}\sqrt{xy}\Big),$$

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Theorem (Hardy's inequality for standard Laguerre functions)

For $\alpha \in [0,\infty)^d$

$$\sum_{n\in\mathbb{N}^d} \frac{|\langle f, \mathcal{L}_n\rangle|}{(|n|+1)^d} \lesssim \|f\|_{H^1(\mathbb{R}^d_+, dx)},$$

uniformly in $f \in H^1(\mathbb{R}^d_+, dx)$. The result is sharp in the sense that for any $\varepsilon > 0$ there exists $f \in H^1(\mathbb{R}^d_+, dx)$ such that

$$\sum_{n\in\mathbb{N}^d} \frac{|\langle f, \mathcal{L}_n\rangle|}{(|n|+1)^{d-\varepsilon}} = \infty.$$

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The Laguerre functions of Hermite type or order $\alpha > -1$ are defined by

$$\varphi_k^\alpha(x) = (2x)^{1/2} \mathcal{L}_k^\alpha(x^2), \qquad x > 0.$$

This system (or rather its multi-dimensional version) is orthonormal and complete in $L^2\left(\mathbb{R}_+^d,\, dx\right)$.

The functions $\{\varphi_n^\alpha\}$ belong to L^∞ if and only if $\alpha \in [-1/2, \infty)^d$. Therefore we restrict to this range of the parameters α.

Similarly as before we can write an explicit formula for the kernels $R_r(x, y)$:

$$R_r^\alpha(x, y) = \frac{2(xy)^{1/2}}{(1-r)r^{\alpha/2}} \exp\left(-\frac{1}{2}\frac{1+r}{1-r}(x^2+y^2)\right) I_\alpha\left(\frac{2r^{1/2}}{1-r}xy\right),$$

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Theorem (Hardy's inequality for Laguerre functions of Hermite type)

For $\alpha \in [-1/2, \infty)^d$

$$\sum_{n \in \mathbb{N}^d} \frac{|\langle f, \varphi_n^\alpha \rangle|}{(|n|+1)^{3d/4}} \lesssim \|f\|_{H^1(\mathbb{R}_+^d, dx)},$$

uniformly in $f \in H^1(\mathbb{R}_+^d, dx)$. The admissible exponent is sharp.

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The one-dimensional *Laguerre functions of convolution type* of order $\alpha > -1$ on $\mathbb{R}_+$ are the functions

$$\ell_k^\alpha(x) = \varphi_k^\alpha(x) x^{-\alpha-1/2}, \qquad x > 0.$$

These functions (for $d \geq 1$) form an orthonormal basis in $L^2(\mathbb{R}_+^d, d\mu_\alpha)$, where

$$d\mu_\alpha(x) = x^{2\alpha+\mathbf{1}}\, dx$$

This measure is doubling and it is known that $N = 2d + 2|\alpha|$, where $|\alpha| = \alpha_1 + \ldots + \alpha_d$.

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Analogously as for Laguerre functions of Hermite type, the explicit formula for $R_r^\alpha(x, y)$ is known and allows us to obtain $\gamma = (|\alpha| + d + 1)/2$.

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uniformly in $f \in H^1(\mathbb{R}_+^d,\, d\mu_\alpha)$. The admissible exponent is sharp.

If the exponent is larger than $d + |\alpha|/2$, then the inequality holds also for $L^1(\mathbb{R}_+^d,\, d\mu_\alpha)$.

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The generalized Hermite functions of order $\lambda \geq 0$ on $\mathbb{R}$ are defined by the relation

$$h_{2k}^{\lambda}(x) = (-1)^k 2^{-1/2} \varphi_k^{\lambda-1/2}(|x|),$$

$$h_{2k+1}^{\lambda}(x) = (-1)^k 2^{-1/2} \operatorname{sgn}(x) \varphi_k^{\lambda+1/2}(|x|),$$

where $x \in \mathbb{R}$ (for $x = 0$ we naturally extend the definition of φ_k^{α}).

Note that (for $d \geq 1$) the functions $\{h_n^{\mathbf{0}}\}_{n \in \mathbb{N}^d}$ are the classical Hermite functions.

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where $x \in \mathbb{R}$ (for $x = 0$ we naturally extend the definition of φ^{α}_k).

Note that (for $d \geq 1$) the functions $\{h^{\mathbf{0}}_n\}_{n \in \mathbb{N}^d}$ are the classical Hermite functions.

The generalized Hermite functions form an orthonormal basis in $L^2(\mathbb{R}^d, dx)$.

The generalized Hermite functions of order $\lambda \geq 0$ on $\mathbb{R}$ are defined by the relation

$$h_{2k}^{\lambda}(x) = (-1)^k 2^{-1/2} \varphi_k^{\lambda-1/2}(|x|),$$

$$h_{2k+1}^{\lambda}(x) = (-1)^k 2^{-1/2} \mathrm{sgn}(x) \varphi_k^{\lambda+1/2}(|x|),$$

where $x \in \mathbb{R}$ (for $x = 0$ we naturally extend the definition of φ_k^{α}).

Note that (for $d \geq 1$) the functions $\{h_n^{\mathbf{0}}\}_{n \in \mathbb{N}^d}$ are the classical Hermite functions.

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Theorem (Hardy's inequality for generalized Hermite functions)

For $\lambda \in [0,\infty)^d$ we have

$$\sum_{n\in\mathbb{N}^d} \frac{|\langle f, h_n^\lambda\rangle|}{(|n|+1)^{3d/4}} \lesssim \|f\|_{H^1(\mathbb{R}^d,dx)},$$

uniformly in $f \in H^1(\mathbb{R}^d, dx)$. The admissible exponent is sharp.

If the exponent is larger than $3d/4$, then the inequality holds also for $L^1(\mathbb{R}^d, dx)$.

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Thank you!