Reduction of integration domain in Triebel–Lizorkin spaces

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Let $1 < p, q < \infty$. We define the **Triebel–Lizorkin** space on a domain $\Omega \subseteq \mathbb{R}^d$ as follows

$$F_{p,q}(\Omega) = \left\{ u \in L^p(\Omega) \colon \int_\Omega \left(\int_\Omega |u(x) - u(y)|^q K(x,y)\, dy \right)^{p/q} dx < \infty \right\}.$$

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Preliminaries

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further called the **full seminorm**.
When Ω is a uniform domain, $K(x,y) = |x-y|^{-d-sq}$, $1 < p, q < \infty$, $0 < s < 1$ and $s > \frac{d}{p} - \frac{d}{q}$, our definition coincides with the classical one, see Prats and Saksman (2017).

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When $p = q$, $s \in (0,1)$, and $K(x,y) = |x-y|^{-d-sq}$ we retrieve the fractional Sobolev space $W^{s,p}(\Omega)$.

The main goal

For $\rho \in (0,1]$ we introduce the **truncated seminorm**:

$$\left(\int_\Omega \left(\int_{B(x,\rho\delta_x)} |u(x) - u(y)|^q K(x,y)\, dy \right)^{p/q} dx \right)^{1/p},$$

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Meta-Theorem

Under certain assumptions on Ω, p, q, and K we have

$$\int_\Omega \left(\int_\Omega |u(x) - u(y)|^q K(x,y)\, dy \right)^{p/q} dx$$
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The results of this type will be called the **comparability** results.

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- Peridynamics – energy functionals with horizons depending on the distance to the boundary, see Q. Du and X. Tian (2017).
- Various modifications of Lévy processes, e.g., the processes with visibility constraint introduced by M. Kassmann and V. Wagner (2018).

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- Prats and Saksman: Ω – bounded uniform domain, $p \geqslant q$, $K(x, y) = |x - y|^{-d-sq}$.
- Kassmann and Wagner: Ω – bounded uniform domain, $p = q = 2$, $k(x, y) \approx \phi(|x - y|)|x - y|^{-d}$. k is a Lévy-type kernel, and ϕ satisfies the weak lower and upper scaling conditions:

$$\lambda^{-\gamma} \lesssim \frac{\phi(\lambda r)}{\phi(r)} \lesssim \lambda^{-\delta}, \quad \lambda \geqslant 1,\ r > 0,$$

  for some constants $0 < \delta \leqslant \gamma < 2$.

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The uniform domains can be strictly defined using the Whitney decomposition $\mathcal{W}$ of Ω: a **chain** from P to Q is a sequence of cubes $(P = R_1, \ldots, R_n = Q)$ such that (the closure of) every cube intersects its successor and predecessor (if it has one).

We define the length of the chain as the sum of the side lengths of its cubes: $\ell(R_1, \ldots, R_n) = \sum \ell(R_i)$.

We also consider the long distance between Q and S:

$D(Q, S) = \ell(Q) + \ell(S) + d(Q, S)$.

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For further reading on uniform domains we refer to Väisälä (1988).

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- In the proof we follow Prats and Saksman.

Lemma

Let Ω be a domain with Whitney covering $\mathcal{W}$, and let ϕ satisfy A1, A2 and A3. Assume that $g \in L^1_{loc}(\mathbb{R}^d)$ and $0 < r < 3\mathrm{diam}(\Omega)$.

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- A2 and A3 together are equivalent to the following upper scaling condition: $\phi(\lambda r) \lesssim \lambda^\gamma \phi(r), \quad \lambda \geqslant 1, r > 0.$

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The 0-order case

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Theorem

Let Ω be a bounded uniform domain. Then, if $1 < q \leqslant p < \infty$, then for every $0 < \rho < 1$

$$\left(\int_\Omega \left(\int_\Omega \frac{|f(x) - f(y)|^q}{|x - y|^d} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

$$\lesssim \left(\int_\Omega \left(\int_{B(x,\rho\delta_x)} \frac{|f(x) - f(y)|^q}{|x - y|^d} (|\log|x - y|| \vee 1) \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

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whereas

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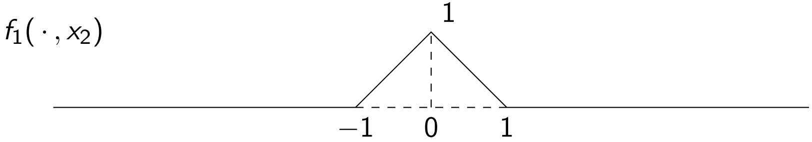

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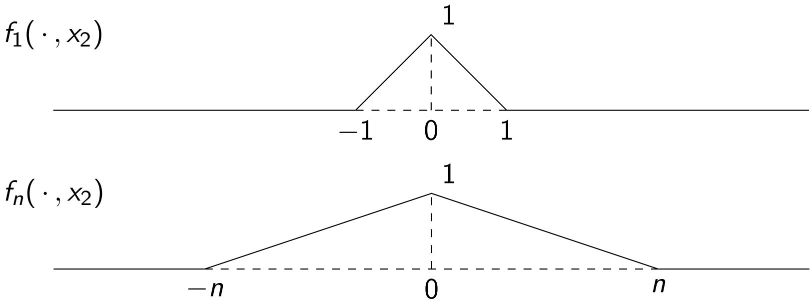

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Theorem

Let $\Omega = \mathbb{R}^k \times (0,1)^l$, $k, l > 0$. Assume that $K(x,y) = |x-y|^{-d-\alpha}$ for $d = k + l$ and $\alpha \in (0,2)$. Then, if

- *$l = 1$ and $\alpha > 1$, or*
- *$l > 1$,*

then the comparability holds.

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