CLT for the capacity of the range of stable random walks

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Introduction

- Let $(S_n)_{n\geq 0}$ be a $\mathbb{Z}^d$-valued random walk.
- The range process is defined as the random set

$$\mathcal{R}_n = \{S_0, S_1, \dots, S_n\}, \quad n \geq 0.$$

Literature overview - cardinality of the range process

- Dvoretzky and Erdös (1950) obtained a law of large numbers for $(\#\mathcal{R}_n)_{n\geq 0}$ when $(S_n)_{n\geq 0}$ is the simple random walk and $d \geq 2$.
- The result was extended by Spitzer for an arbitrary random walk in $d \geq 1$.
- CLT for $(\#\mathcal{R}_n)_{n\geq 0}$ was obtained by Jain and Orey (1968) when $(S_n)_{n\geq 0}$ is strongly transient.

Assumptions on $(S_n)_{n\geq 0}$

- $(S_n)_{n\geq 0}$ is called transient if

$$\sum_{n\geq 0} p_n(0) < \infty.$$

- $(S_n)_{n\geq 0}$ is called strongly transient if

$$\sum_{n\geq 0} np_n(0) < \infty.$$

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- The result was extended by Spitzer for an arbitrary random walk in $d \geq 1$.
- CLT for $(\#\mathcal{R}_n)_{n\geq 0}$ was obtained by Jain and Orey (1968) when $(S_n)_{n\geq 0}$ is strongly transient.
- CLT for $(\#\mathcal{R}_n)_{n\geq 0}$ in dimensions 3 and 4 was obtained by Jain and Pruitt (1969).
- Le Gall and Rosen (1991) were the first who considered the strong law of large numbers and the central limit theorem for $(\#\mathcal{R}_n)_{n\geq 0}$ in the case when $(S_n)_{n\geq 0}$ is a stable aperiodic random walk.

Introduction

- The capacity of a set $A \subseteq \mathbb{Z}^d$ (with respect to $(S_n)_{n\geq 0}$) is defined as

$$\mathrm{Cap}(A) = \sum_{x\in A} \mathbb{P}_x(T_A^+ = \infty),$$

where T_A^+ denotes the first return time of $(S_n)_{n\geq 0}$ to the set A, that is

$$T_A^+ = \inf\{n \geq 1 : S_n \in A\}.$$

- Define the process $(\mathcal{C}_n)_{n\geq 0}$ as

$$\mathcal{C}_n = \mathrm{Cap}(\mathcal{R}_n).$$

- The main aim is to establish a central limit theorem for the process $(\mathcal{C}_n)_{n\geq 0}$.

Literature overview - capacity of the range process

- The first results on the long-time behavior of the capacity process $(\mathcal{C}_n)_{n\geq 0}$ are due to Jain and Orey (Israel J. Math. 1968) where they obtained a version of the strong law of large numbers for any transient random walk.
- Recently Asselah, Schapira and Sousi (TAMS, 2018) proved a central limit theorem for $(\mathcal{C}_n)_{n\geq 0}$ for the simple random walk in $d \geq 6$.
- The same authors proved versions of a law of large numbers and central limit theorem in the case $d = 4$ (Ann. Probab., 2019).
- Less than one month ago Schapira proved a central limit theorem for $(\mathcal{C}_n)_{n\geq 0}$ in $d = 5$ for symmetric and irreducible random walks with finite d-th moment.

Assumptions on $(S_n)_{n\geq 0}$

(A1) $(S_n)_{n\geq 0}$ is aperiodic, that is, the smallest additive subgroup generated by the set $\operatorname{supp} p_1 = \{x \in \mathbb{Z}^d : p_1(x) > 0\}$ is equal to $\mathbb{Z}^d$.

(A2) $(S_n)_{n\geq 0}$ is symmetric and strongly transient.

(A3) $(S_n)_{n\geq 0}$ belongs to the domain of attraction of a non-degenerate α-stable random law with $0 < \alpha \leq 2$, meaning that there exists a regularly varying function $b(x)$ with index $1/\alpha$ such that

$$\frac{S_n}{b(n)} \xrightarrow[n\nearrow\infty]{\text{(d)}} X_\alpha,$$

where X_α is an α-stable random variable on $\mathbb{R}^d$ and $\xrightarrow{\text{(d)}}$ stands for the convergence in distribution.

(A4) $(S_n)_{n\geq 0}$ admits one-step loops, that is, $p = p_1(0) > 0$.

Assumptions on $(S_n)_{n\geq 0}$

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$$\sum_{n\geq 0} p_n(0) < \infty.$$

- $(S_n)_{n\geq 0}$ is called strongly transient if

$$\sum_{n\geq 0} np_n(0) < \infty.$$

- Under (**A3**), $(S_n)_{n\geq 0}$ is transient if $d > \alpha$ and strongly transient if $d > 2\alpha$.

Examples (subordinate random walks)

- Let $(Z_n)_{n\geq 0}$ be a simple symmetric random walk in $\mathbb{Z}^d$.
- Let $(\tau_n)_{n\geq 0}$ be an increasing random walk in $\mathbb{Z}$ starting from 0 that is independent of $(Z_n)_{n\geq 0}$ and which is uniquely determined with

$$\mathbb{E}[e^{-\lambda\tau_1}] = 1 - \psi(1 - e^{-\lambda}),$$

  where $\psi(\lambda)$ is a Bernstein function with $\psi(0) = 0$ and $\psi(1) = 1$.
- We define the subordinate random walk as $S_n = Z_{\tau_n}$, $n \geq 0$.
- Such walks satisfy (**A3**) with index $0 < \alpha \leq 2$ if and only if the function $\psi(\lambda)$ is regularly varying at zero with index $\alpha/2$ (A. Mimica, *On subordinate random walks*, Forum Math., 2017).

CLT for $(\mathcal{C}_n)_{n\geq 0}$

Theorem 2.1

*Assume **(A1)**-**(A4)** and $d \geq 3\alpha$. Then, there is a constant $\sigma_d > 0$ such that*

$$\frac{\mathcal{C}_n - \mathbb{E}[\mathcal{C}_n]}{\sqrt{n}} \xrightarrow[n\nearrow\infty]{(d)} \sigma_d\,\mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ stands for the standard normal distribution.

On the SLLN for $(\mathcal{C}_n)_{n\geq 0}$

Theorem 3.1

*Under (**A2**) there exists* $\mu_d > 0$ *such that*

$$\lim_{n\nearrow\infty} \frac{\mathcal{C}_n}{n} = \mu_d \quad \mathbb{P}\text{-a.s.}$$

Inequality for capacities

Proposition 4.1 (Asselah, Schapira and Sousi)

Let A and B be finite subsets of $\mathbb{Z}^d$. Then,

$$\mathrm{Cap}(A \cup B) \geq \mathrm{Cap}(A) + \mathrm{Cap}(B) - 2G(A, B),$$

where G is the Green function of $(S_n)_{n\geq 0}$ and

$$G(A, B) = \sum_{x \in A} \sum_{y \in B} G(x, y).$$

Capacity decomposition

- Since the capacity is translation invariant ($\mathrm{Cap}(A+x) = \mathrm{Cap}(A)$ for all $x \in \mathbb{Z}^d$), we have

$$\begin{aligned}\mathrm{Cap}(\mathcal{R}_n) &= \mathrm{Cap}(\mathcal{R}_{n/2} \cup \mathcal{R}[n/2, n]) \\ &= \mathrm{Cap}((\mathcal{R}_{n/2} \cup \mathcal{R}[n/2, n]) - S_{n/2}) \\ &= \mathrm{Cap}((\mathcal{R}_{n/2} - S_{n/2}) \cup (\mathcal{R}[n/2, n] - S_{n/2})).\end{aligned}$$

- By the Markov property, the random variables $\mathcal{C}_{n/2}^{(1)} = \mathrm{Cap}(\mathcal{R}_{n/2} - S_{n/2})$ and $\mathcal{C}_{n/2}^{(2)} = \mathrm{Cap}(\mathcal{R}[n/2, n] - S_{n/2})$ are independent and, by reversibility, each of them has the same law as $\mathcal{C}_{n/2}$ or $\mathcal{C}_{n/2+1}$.

Capacity decomposition

- Applying Proposition 4.1 we get

$$\mathcal{C}_n \geq \mathcal{C}^{(1)}_{n/2} + \mathcal{C}^{(2)}_{n/2} - 2 \sum_{x \in \mathcal{R}^{(1)}_{n/2}} \sum_{y \in \mathcal{R}^{(2)}_{n/2}} G(x,y).$$

- Applying the same subdivision to each of the terms $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ and iterating L times ($2^L \leq n$), we obtain the dyadic decomposition

$$\mathcal{C}_n \geq \sum_{i=1}^{2^L} \mathrm{Cap}\left(\mathcal{R}^{(i)}_{n/2^L}\right) - 2 \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathcal{E}^{(i)}_l,$$

where $\mathcal{E}^{(i)}_l$ has the same law as $\sum_{x \in \mathcal{R}_{n/2^l}} \sum_{y \in \mathcal{R}'_{n/2^l}} G(x,y)$, with $\mathcal{R}'$ independent copy of $\mathcal{R}$.

Error term estimates

Lemma 4.2

*Assume (**A1**)-(**A3**). Let $(S'_n)_{n\geq 0}$ be an independent copy of $(S_n)_{n\geq 0}$ and denote the corresponding range process by $(\mathcal{R}'_n)_{n\geq 0}$. Then, for all $k, n \geq 1$ we have that*

$$\mathbb{E}\left[G(\mathcal{R}_n, \mathcal{R}'_n)^k\right] \leq C(h_d(n))^k,$$

where $C > 0$ is a constant that depends only on k, and $h_d(n)$ is given by

$$h_d(n) = \begin{cases} 1, & d/\alpha > 3, \\ \sum_{k=1}^{n} k^{-1}\ell(k)^{-d}, & d/\alpha = 3, \\ n^3(b(n))^{-d}, & 2 < d/\alpha < 3, \end{cases}$$

where ℓ is a slowly varying function.

This result is in the spirit of the estimates of moments of intersection times of random walks.

Variance estimates

Lemma 4.3

*Assume (**A1**)-(**A3**) and* $d \geq 3\alpha$. *Then the sequence* $(\mathrm{Var}(\mathcal{C}_n)/n)_{n\geq 0}$ *converges to* $\sigma_d \geq 0$.

Lemma 4.4

*Assume (**A1**)-(**A4**) and* $d \geq 3\alpha$. *Then* $\sigma_d > 0$.

Proof of CLT

- By the dyadic decomposition and subadditivity of the capacity, we get

$$\sum_{i=1}^{2^L} \mathrm{Cap}\left(\mathcal{R}^{(i)}_{n/2^L}\right) - 2\sum_{l=1}^{L}\sum_{i=1}^{2^{l-1}} \mathcal{E}^{(i)}_l \le \mathcal{C}_n \le \sum_{i=1}^{2^L} \mathrm{Cap}\left(\mathcal{R}^{(i)}_{n/2^L}\right).$$

- Denote $\mathcal{C}^{(i)}_{n/2^L} = \mathrm{Cap}(\mathcal{R}^{(i)}_{n/2^L})$, $i = 1, 2, \dots, L$. By taking expectation and then subtracting, we obtain (using notation $\overline{\mathcal{C}}_n = \mathcal{C}_n - \mathbb{E}[\mathcal{C}_n]$)

$$\sum_{i=1}^{2^L} \overline{\mathcal{C}}^{(i)}_{n/2^L} - 2\sum_{l=1}^{L}\sum_{i=1}^{2^{l-1}} \mathcal{E}^{(i)}_l \le \overline{\mathcal{C}}_n \le \sum_{i=1}^{2^L} \overline{\mathcal{C}}^{(i)}_{n/2^L} + 2\sum_{l=1}^{L}\sum_{i=1}^{2^{l-1}} \mathbb{E}[\mathcal{E}^{(i)}_l].$$

Proof of CLT

- We define
$$\mathcal{E}(n) = \sum_{i=1}^{2^L} \overline{\mathcal{C}}^{(i)}_{n/2^L} - \overline{\mathcal{C}}_n.$$
- We first show
$$\lim_{n \nearrow \infty} \frac{\mathbb{E}[|\mathcal{E}(n)|]}{\sqrt{n}} = 0.$$
- In the end we show
$$\sum_{i=1}^{2^L} \frac{\overline{\mathcal{C}}^{(i)}_{n/2^L}}{\sqrt{n}} \xrightarrow[n \nearrow \infty]{\text{(d)}} \sigma_d \, \mathcal{N}(0,1)$$

  using Lindeberg-Feller central limit theorem.

Thank you for your attention :)