On the maximal operator of an arbitrary Ornstein-Uhlenbeck semigroup

Peter Sjögren
University of Gothenburg

Joint work with Valentina Casarino and Paolo Ciatti (Padova)

In $\mathbb{R}^n$ we will consider the semigroup generated by the operator L defined by

In $\mathbb{R}^n$ we will consider the semigroup generated by the operator L defined by

$$L = \frac{1}{2} \sum_{i,j} q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j} b_{ij} x_j \frac{\partial}{\partial x_i}$$

In $\mathbb{R}^n$ we will consider the semigroup generated by the operator L defined by

$$L = \frac{1}{2} \sum_{i,j} q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j} b_{ij} x_j \frac{\partial}{\partial x_i}$$

or equivalently

$$Lf = \frac{1}{2} \operatorname{tr}(Q \nabla^2 f) + \langle Bx, \nabla f \rangle.$$

In $\mathbb{R}^n$ we will consider the semigroup generated by the operator L defined by

$$L = \frac{1}{2} \sum_{i,j} q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j} b_{ij} x_j \frac{\partial}{\partial x_i}$$

or equivalently

$$Lf = \frac{1}{2} \operatorname{tr}(Q \nabla^2 f) + \langle Bx, \nabla f \rangle.$$

Here Q is a real, symmetric and positive definite matrix.

In $\mathbb{R}^n$ we will consider the semigroup generated by the operator L defined by

$$L = \frac{1}{2} \sum_{i,j} q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j} b_{ij} x_j \frac{\partial}{\partial x_i}$$

or equivalently

$$Lf = \frac{1}{2} \operatorname{tr}(Q \nabla^2 f) + \langle Bx, \nabla f \rangle.$$

Here Q is a real, symmetric and positive definite matrix.

B is a real matrix which is negative in the sense that all its eigenvalues have negative real parts.

In $\mathbb{R}^n$ we will consider the semigroup generated by the operator L defined by

$$L = \frac{1}{2} \sum_{i,j} q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j} b_{ij} x_j \frac{\partial}{\partial x_i}$$

or equivalently

$$Lf = \frac{1}{2} \operatorname{tr}(Q \nabla^2 f) + \langle Bx, \nabla f \rangle.$$

Here Q is a real, symmetric and positive definite matrix.

B is a real matrix which is negative in the sense that all its eigenvalues have negative real parts.

Q indicates the covariance and B the drift of the operator.

The semigroup is formally $\mathcal{H}_t = e^{tL}$, $t > 0$,

The semigroup is formally $\mathcal{H}_t = e^{tL}$, $t > 0$, but to define it we first introduce the positive definite, symmetric matrices

$$Q_t = \int_0^t e^{sB}\, Q\, e^{sB^*}\, ds, \qquad 0 < t \leq \infty,$$

The semigroup is formally $\mathcal{H}_t = e^{tL}$, $t > 0$, but to define it we first introduce the positive definite, symmetric matrices

$$Q_t = \int_0^t e^{sB}\, Q\, e^{sB^*}\, ds, \qquad 0 < t \leq \infty,$$

and the normalized gaussian measures in $\mathbb{R}^n$

$$d\gamma_t(y) = (2\pi)^{-\frac{n}{2}}\, (\det Q_t)^{-\frac{1}{2}}\, \exp\left(-\frac{1}{2}\langle Q_t^{-1} y, y\rangle\right) dy.$$

The semigroup is formally $\mathcal{H}_t = e^{tL},\ t > 0,$ but to define it we first introduce the positive definite, symmetric matrices

$$Q_t = \int_0^t e^{sB}\, Q\, e^{sB^*}\, ds, \qquad 0 < t \leq \infty,$$

and the normalized gaussian measures in $\mathbb{R}^n$

$$d\gamma_t(y) = (2\pi)^{-\frac{n}{2}}\, (\det Q_t)^{-\frac{1}{2}}\, \exp\left(-\frac{1}{2}\langle Q_t^{-1} y, y\rangle\right) dy.$$

Then one defines

$$\mathcal{H}_t f(x) = \int f(e^{tB}x - y)\, d\gamma_t(y),$$

The semigroup is formally $\mathcal{H}_t = e^{tL}$, $t > 0$, but to define it we first introduce the positive definite, symmetric matrices

$$Q_t = \int_0^t e^{sB}\, Q\, e^{sB^*}\, ds, \qquad 0 < t \leq \infty,$$

and the normalized gaussian measures in $\mathbb{R}^n$

$$d\gamma_t(y) = (2\pi)^{-\frac{n}{2}}\, (\det Q_t)^{-\frac{1}{2}}\, \exp\left(-\frac{1}{2}\langle Q_t^{-1} y, y\rangle\right) dy.$$

Then one defines

$$\mathcal{H}_t f(x) = \int f(e^{tB}x - y)\, d\gamma_t(y),$$

a formula due to Kolmogorov.

The semigroup is formally $\mathcal{H}_t = e^{tL}$, $t > 0$, but to define it we first introduce the positive definite, symmetric matrices

$$Q_t = \int_0^t e^{sB}\, Q\, e^{sB^*}\, ds, \qquad 0 < t \leq \infty,$$

and the normalized gaussian measures in $\mathbb{R}^n$

$$d\gamma_t(y) = (2\pi)^{-\frac{n}{2}}\, (\det Q_t)^{-\frac{1}{2}}\, \exp\left(-\frac{1}{2}\langle Q_t^{-1} y, y\rangle\right) dy.$$

Then one defines

$$\mathcal{H}_t f(x) = \int f(e^{tB}x - y)\, d\gamma_t(y),$$

a formula due to Kolmogorov.

The measure γ_∞ is invariant under the action of $\mathcal{H}_t$.

The semigroup is formally $\mathcal{H}_t = e^{tL}$, $t > 0$, but to define it we first introduce the positive definite, symmetric matrices

$$Q_t = \int_0^t e^{sB}\, Q\, e^{sB^*}\, ds, \qquad 0 < t \le \infty,$$

and the normalized gaussian measures in $\mathbb{R}^n$

$$d\gamma_t(y) = (2\pi)^{-\frac{n}{2}} \, (\det Q_t)^{-\frac{1}{2}} \, \exp\left(-\frac{1}{2}\langle Q_t^{-1} y, y\rangle\right) dy.$$

Then one defines

$$\mathcal{H}_t f(x) = \int f(e^{tB}x - y)\, d\gamma_t(y),$$

a formula due to Kolmogorov.

The measure γ_∞ is invariant under the action of $\mathcal{H}_t$.
It will be our basic measure, replacing Lebesgue measure.

We **remark** that $(\mathcal{H}_t)_{t>0}$ is the transition semigroup of the stochastic process

$$\chi(x,t) = e^{tB}x + \int_0^t e^{(t-s)B}\,dW(s),$$

We **remark** that $(\mathcal{H}_t)_{t>0}$ is the transition semigroup of the stochastic process

$$\chi(x,t) = e^{tB}x + \int_0^t e^{(t-s)B}\,dW(s),$$

where W is a Brownian motion in $\mathbb{R}^n$ with covariance matrix Q.

We **remark** that $(\mathcal{H}_t)_{t>0}$ is the transition semigroup of the stochastic process

$$\chi(x,t) = e^{tB}x + \int_0^t e^{(t-s)B}\, dW(s),$$

where W is a Brownian motion in $\mathbb{R}^n$ with covariance matrix Q.

This process describes the random motion of a particle subject to friction.

Standard case: $Q = I$ and $B = -I$

Standard case: $Q = I$ and $B = -I$

Here $d\gamma_\infty(y) = \pi^{-\frac{n}{2}} e^{-|y|^2} dy$ is the standard gaussian measure,

Standard case: $Q = I$ and $B = -I$

Here $d\gamma_\infty(y) = \pi^{-\frac{n}{2}} e^{-|y|^2} dy$ is the standard gaussian measure, and

$$\mathcal{H}_t f(x) = \frac{e^{|x|^2}}{(1-e^{-2t})^{n/2}} \int f(y) \exp\left(-\frac{|x-e^{-t}y|^2}{1-e^{-2t}}\right) d\gamma_\infty(y).$$

Standard case: $Q = I$ and $B = -I$

Here $d\gamma_\infty(y) = \pi^{-\frac{n}{2}} e^{-|y|^2} dy$ is the standard gaussian measure, and

$$\mathcal{H}_t f(x) = \frac{e^{|x|^2}}{(1 - e^{-2t})^{n/2}} \int f(y) \exp\left(-\frac{|x - e^{-t}y|^2}{1 - e^{-2t}}\right) d\gamma_\infty(y).$$

This is the classical Mehler kernel.

Results, in the general case

Results, in the general case

As often for semigroups, one is interested in the pointwise convergence $\mathcal{H}_t f \to f$ as $t \to 0$

Results, in the general case

As often for semigroups, one is interested in the pointwise convergence $\mathcal{H}_t f \to f$ as $t \to 0$ and thus in the maximal operator

$$\mathcal{H}_* f(x) = \sup_{t>0} \left| \mathcal{H}_t f(x) \right|.$$

Results, in the general case

As often for semigroups, one is interested in the pointwise convergence $\mathcal{H}_t f \to f$ as $t \to 0$ and thus in the maximal operator

$$\mathcal{H}_* f(x) = \sup_{t>0} \left| \mathcal{H}_t f(x) \right|.$$

Main result:

Theorem

The operator $\mathcal{H}_*$ is of weak type (1,1) with respect to the invariant measure γ_∞.

Results, in the general case

As often for semigroups, one is interested in the pointwise convergence $\mathcal{H}_t f \to f$ as $t \to 0$ and thus in the maximal operator

$$\mathcal{H}_* f(x) = \sup_{t>0} \left|\mathcal{H}_t f(x)\right|.$$

Main result:

Theorem

The operator $\mathcal{H}_*$ is of weak type (1,1) with respect to the invariant measure γ_∞.

Corollary

$\mathcal{H}_*$ is bounded on $L^p(\gamma_\infty)$ for $1 < p \leq \infty$.

History

History

L^p:

History

L^p:

The boundedness of $\mathcal{H}_*$ on L^p, $p > 1$, follows by general Littlewood–Paley–Stein theory,

History

L^p:

The boundedness of $\mathcal{H}_*$ on $L^p,\ p > 1$, follows by general Littlewood–Paley–Stein theory, when the $\mathcal{H}_t,\ t > 0$ are symmetric on $L^2(\gamma_\infty)$.

History

L^p:

The boundedness of $\mathcal{H}_*$ on $L^p,\ p > 1,$ follows by general Littlewood–Paley–Stein theory, when the $\mathcal{H}_t,\ t > 0$ are symmetric on $L^2(\gamma_\infty)$.

Mauceri and Noselli extended this in 2009 to the case when the $\mathcal{H}_t$ are only assumed normal on $L^2(\gamma_\infty)$.

Weak type (1,1):

Weak type (1,1):

In the standard case,

Weak type (1,1):

In the standard case, the weak type (1,1) of $\mathcal{H}_*$ was first proved in dimension 1 by Muckenhoupt in 1969,

Weak type (1,1):

In the standard case, the weak type (1,1) of $\mathcal{H}_*$ was first proved in dimension 1 by Muckenhoupt in 1969,
and then in any finite dimension by S. in 1983.

Weak type (1,1):

In the standard case, the weak type (1,1) of $\mathcal{H}_*$ was first proved in dimension 1 by Muckenhoupt in 1969,
and then in any finite dimension by S. in 1983.

Mauceri and Noselli proved it under a rather technical assumption, which implies that all the $\mathcal{H}_t$ are normal operators.

Weak type (1,1):

In the standard case, the weak type (1,1) of $\mathcal{H}_*$ was first proved in dimension 1 by Muckenhoupt in 1969,
and then in any finite dimension by S. in 1983.

Mauceri and Noselli proved it under a rather technical assumption, which implies that all the $\mathcal{H}_t$ are normal operators.

In 2017, the present authors proved the the weak type (1,1), assuming only the $\mathcal{H}_t$ normal.

Weak type (1,1):

In the standard case, the weak type (1,1) of $\mathcal{H}_*$ was first proved in dimension 1 by Muckenhoupt in 1969,
and then in any finite dimension by S. in 1983.

Mauceri and Noselli proved it under a rather technical assumption, which implies that all the $\mathcal{H}_t$ are normal operators.

In 2017, the present authors proved the the weak type (1,1), assuming only the $\mathcal{H}_t$ normal. The proof is based on a reduction to the situation when $Q = I$ and B consists of 2×2 blocks along the diagonal.

The first step of the proof of our result in the **general case** is to write the Mehler kernel.

The first step of the proof of our result in the **general case** is to write the Mehler kernel.

We introduce the quadratic form

$$R(x) = \frac{1}{2}\left\langle Q_\infty^{-1}x, x\right\rangle = \frac{1}{2}\left|Q_\infty^{-1/2}x\right|^2, \qquad x \in \mathbb{R}^n.$$

The first step of the proof of our result in the **general case** is to write the Mehler kernel.

We introduce the quadratic form

$$R(x) = \frac{1}{2}\left\langle Q_\infty^{-1}x, x\right\rangle = \frac{1}{2}\left|Q_\infty^{-1/2}x\right|^2, \qquad x \in \mathbb{R}^n.$$

so that $d\gamma_\infty(x) = e^{R(x)}\,dx$.

The first step of the proof of our result in the **general case** is to write the Mehler kernel.

We introduce the quadratic form

$$R(x) = \frac{1}{2}\left\langle Q_\infty^{-1} x, x\right\rangle = \frac{1}{2}\left|Q_\infty^{-1/2} x\right|^2, \qquad x \in \mathbb{R}^n.$$

so that $d\gamma_\infty(x) = e^{R(x)}\, dx$.

The form $R(x)$ is positive definite,

The first step of the proof of our result in the **general case** is to write the Mehler kernel.

We introduce the quadratic form

$$R(x) = \frac{1}{2}\left\langle Q_\infty^{-1}x, x\right\rangle = \frac{1}{2}\left|Q_\infty^{-1/2}x\right|^2, \qquad x \in \mathbb{R}^n.$$

so that $d\gamma_\infty(x) = e^{R(x)}\,dx$.

The form $R(x)$ is positive definite,
so there is a family of ellipsoids defined by

$$E_\beta = \{x \in \mathbb{R}^n : \ R(x) = \beta\}, \qquad \beta > 0.$$

The first step of the proof of our result in the **general case** is to write the Mehler kernel.

We introduce the quadratic form

$$R(x) = \frac{1}{2}\left\langle Q_\infty^{-1}x, x\right\rangle = \frac{1}{2}\left|Q_\infty^{-1/2}x\right|^2, \qquad x \in \mathbb{R}^n.$$

so that $d\gamma_\infty(x) = e^{R(x)}\,dx$.

The form $R(x)$ is positive definite,
so there is a family of ellipsoids defined by

$$E_\beta = \{x \in \mathbb{R}^n : \; R(x) = \beta\}, \qquad \beta > 0.$$

We also introduce a one-parameter group of matrices

The first step of the proof of our result in the **general case** is to write the Mehler kernel.

We introduce the quadratic form

$$R(x) = \frac{1}{2}\left\langle Q_\infty^{-1} x, x \right\rangle = \frac{1}{2}\left| Q_\infty^{-1/2} x \right|^2, \qquad x \in \mathbb{R}^n.$$

so that $d\gamma_\infty(x) = e^{R(x)}\, dx$.

The form $R(x)$ is positive definite,
so there is a family of ellipsoids defined by

$$E_\beta = \{x \in \mathbb{R}^n : \; R(x) = \beta\}, \qquad \beta > 0.$$

We also introduce a one-parameter group of matrices

$$D_t = Q_\infty \, e^{-tB^*} \, Q_\infty^{-1}, \qquad t \in \mathbb{R}.$$

The first step of the proof of our result in the **general case** is to write the Mehler kernel.

We introduce the quadratic form

$$R(x) = \frac{1}{2}\left\langle Q_\infty^{-1} x, x \right\rangle = \frac{1}{2}\left| Q_\infty^{-1/2} x \right|^2, \qquad x \in \mathbb{R}^n.$$

so that $d\gamma_\infty(x) = e^{R(x)}\, dx$.

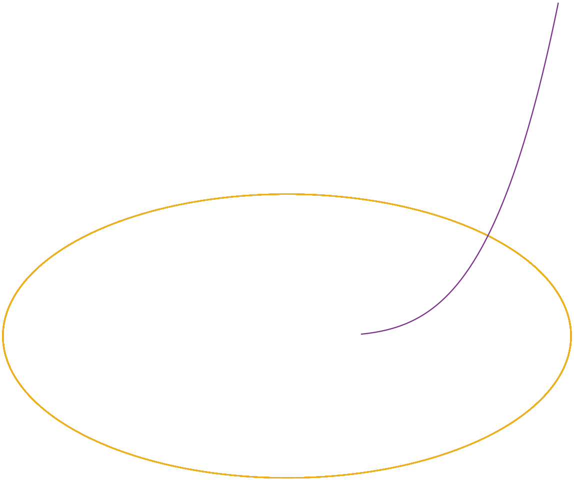
The form $R(x)$ is positive definite,
so there is a family of ellipsoids defined by

$$E_\beta = \{x \in \mathbb{R}^n : \; R(x) = \beta\}, \qquad \beta > 0.$$

We also introduce a one-parameter group of matrices

$$D_t = Q_\infty \, e^{-tB^*} \, Q_\infty^{-1}, \qquad t \in \mathbb{R}.$$

The following figure shows an example of an ellipsoid E_β and a trajectory $D_t \, \widetilde{x}, \;\; t \in \mathbb{R}$.



Now fix $\beta > 0$ and E_β.

Now fix $\beta > 0$ and E_β. Any point $x \neq 0$ can be written uniquely as $x = D_t \widetilde{x}$ with $\widetilde{x} \in E_\beta$,

Now fix $\beta > 0$ and E_β. Any point $x \neq 0$ can be written uniquely as $x = D_t \widetilde{x}$ with $\widetilde{x} \in E_\beta$,
and we can use $(\widetilde{x}, t)$ as polar coordinates for x.

Now fix $\beta > 0$ and E_β. Any point $x \neq 0$ can be written uniquely as $x = D_t \widetilde{x}$ with $\widetilde{x} \in E_\beta$,
and we can use $(\widetilde{x}, t)$ as polar coordinates for x.

After some computations, starting from Kolmogorov's formula,

Now fix $\beta > 0$ and E_β. Any point $x \neq 0$ can be written uniquely as $x = D_t \widetilde{x}$ with $\widetilde{x} \in E_\beta$,
and we can use $(\widetilde{x}, t)$ as polar coordinates for x.

After some computations, starting from Kolmogorov's formula, one finds

$$\mathcal{H}_t f(x) = \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,,$$

Now fix $\beta > 0$ and E_β. Any point $x \neq 0$ can be written uniquely as $x = D_t\,\widetilde{x}$ with $\widetilde{x} \in E_\beta$,
and we can use $(\widetilde{x}, t)$ as polar coordinates for x.

After some computations, starting from Kolmogorov's formula, one finds

$$\mathcal{H}_t f(x) = \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,,$$

where the Mehler kernel K_t is given by

Now fix $\beta > 0$ and E_β. Any point $x \neq 0$ can be written uniquely as $x = D_t \widetilde{x}$ with $\widetilde{x} \in E_\beta$,
and we can use $(\widetilde{x}, t)$ as polar coordinates for x.

After some computations, starting from Kolmogorov's formula, one finds

$$\mathcal{H}_t f(x) = \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,,$$

where the Mehler kernel K_t is given by

$$K_t(x,u) =$$

$$\left(\frac{\det Q_\infty}{\det Q_t}\right)^{1/2} e^{R(x)} \exp\left[-\frac{1}{2}\left\langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t x)\,,\ u - D_t x\right\rangle\right].$$

We will need to integrate the Gaussian measure in terms of the polar coordinates.

We will need to integrate the Gaussian measure in terms of the polar coordinates.
In particular, we need to know that the curves $t \mapsto D_t v$ are transversal to the ellipsoids E_β,

We will need to integrate the Gaussian measure in terms of the polar coordinates.
In particular, we need to know that the curves $t \mapsto D_t v$ are transversal to the ellipsoids E_β, which are the level surfaces of the quadratic form $R(x) = \frac{1}{2} \left\langle Q_\infty^{-1/2} x, Q_\infty^{-1/2} x \right\rangle$.

We will need to integrate the Gaussian measure in terms of the polar coordinates.
In particular, we need to know that the curves $t \mapsto D_t v$ are transversal to the ellipsoids E_β, which are the level surfaces of the quadratic form $R(x) = \frac{1}{2}\left\langle Q_\infty^{-1/2}x, Q_\infty^{-1/2}x\right\rangle$.

So we compute $\partial_t R(D_t v)$ for $v \neq 0$, to see that it is positive.

We will need to integrate the Gaussian measure in terms of the polar coordinates.
In particular, we need to know that the curves $t \mapsto D_t v$ are transversal to the ellipsoids E_β, which are the level surfaces of the quadratic form $R(x) = \frac{1}{2}\left\langle Q_\infty^{-1/2}x, Q_\infty^{-1/2}x\right\rangle$.

So we compute $\partial_t R(D_t v)$ for $v \neq 0$, to see that it is positive.
It equals

We will need to integrate the Gaussian measure in terms of the polar coordinates.
In particular, we need to know that the curves $t \mapsto D_t v$ are transversal to the ellipsoids E_β, which are the level surfaces of the quadratic form $R(x) = \frac{1}{2}\left\langle Q_\infty^{-1/2} x, Q_\infty^{-1/2} x \right\rangle$.

So we compute $\partial_t R(D_t v)$ for $v \neq 0$, to see that it is positive.
It equals

$$\frac{1}{2}\frac{\partial}{\partial t}\left\langle Q_\infty^{-1/2} D_t v,\ Q_\infty^{-1/2} D_t v \right\rangle = \left\langle Q_\infty^{-1/2} \frac{\partial}{\partial t} D_t v,\ Q_\infty^{-1/2} D_t v \right\rangle .$$

We will need to integrate the Gaussian measure in terms of the polar coordinates.
In particular, we need to know that the curves $t \mapsto D_t v$ are transversal to the ellipsoids E_β, which are the level surfaces of the quadratic form $R(x) = \frac{1}{2}\left\langle Q_\infty^{-1/2} x, Q_\infty^{-1/2} x\right\rangle$.

So we compute $\partial_t R(D_t v)$ for $v \neq 0$, to see that it is positive.
It equals

$$\frac{1}{2}\frac{\partial}{\partial t}\left\langle Q_\infty^{-1/2} D_t v,\ Q_\infty^{-1/2} D_t v\right\rangle = \left\langle Q_\infty^{-1/2}\frac{\partial}{\partial t} D_t v,\ Q_\infty^{-1/2} D_t v\right\rangle .$$

Here, since $D_t = Q_\infty e^{-tB^*} Q_\infty^{-1}$,

We will need to integrate the Gaussian measure in terms of the polar coordinates.
In particular, we need to know that the curves $t \mapsto D_t v$ are transversal to the ellipsoids E_β, which are the level surfaces of the quadratic form $R(x) = \frac{1}{2}\left\langle Q_\infty^{-1/2} x, Q_\infty^{-1/2} x \right\rangle$.

So we compute $\partial_t R(D_t v)$ for $v \neq 0$, to see that it is positive.
It equals

$$\frac{1}{2}\frac{\partial}{\partial t}\left\langle Q_\infty^{-1/2} D_t v,\ Q_\infty^{-1/2} D_t v \right\rangle = \left\langle Q_\infty^{-1/2} \frac{\partial}{\partial t} D_t v,\ Q_\infty^{-1/2} D_t v \right\rangle .$$

Here, since $D_t = Q_\infty e^{-tB^*} Q_\infty^{-1}$,

$$\frac{\partial}{\partial t} D_t v = -Q_\infty B^* e^{-tB^*} Q_\infty^{-1} v = -Q_\infty B^* Q_\infty^{-1} D_t v.$$

We will need to integrate the Gaussian measure in terms of the polar coordinates.
In particular, we need to know that the curves $t \mapsto D_t v$ are transversal to the ellipsoids E_β, which are the level surfaces of the quadratic form $R(x) = \frac{1}{2}\left\langle Q_\infty^{-1/2}x, Q_\infty^{-1/2}x\right\rangle$.

So we compute $\partial_t R(D_t v)$ for $v \neq 0$, to see that it is positive.
It equals

$$\frac{1}{2}\frac{\partial}{\partial t}\left\langle Q_\infty^{-1/2}D_t v,\ Q_\infty^{-1/2}D_t v\right\rangle = \left\langle Q_\infty^{-1/2}\frac{\partial}{\partial t}D_t v,\ Q_\infty^{-1/2}D_t v\right\rangle.$$

Here, since $D_t = Q_\infty e^{-tB^*} Q_\infty^{-1}$,

$$\frac{\partial}{\partial t}D_t v = -Q_\infty B^* e^{-tB^*} Q_\infty^{-1} v = -Q_\infty B^* Q_\infty^{-1} D_t v.$$

Writing $w = Q_\infty^{-1} D_t v$, we get

We will need to integrate the Gaussian measure in terms of the polar coordinates.
In particular, we need to know that the curves $t \mapsto D_t\, v$ are transversal to the ellipsoids E_β, which are the level surfaces of the quadratic form $R(x) = \frac{1}{2}\left\langle Q_\infty^{-1/2} x, Q_\infty^{-1/2} x \right\rangle$.

So we compute $\partial_t R(D_t\, v)$ for $v \neq 0$, to see that it is positive.
It equals

$$\frac{1}{2}\frac{\partial}{\partial t}\left\langle Q_\infty^{-1/2} D_t\, v,\ Q_\infty^{-1/2} D_t\, v \right\rangle = \left\langle Q_\infty^{-1/2} \frac{\partial}{\partial t} D_t\, v,\ Q_\infty^{-1/2} D_t\, v \right\rangle .$$

Here, since $D_t = Q_\infty\, e^{-tB^*}\, Q_\infty^{-1}$,

$$\frac{\partial}{\partial t} D_t\, v = -Q_\infty B^* e^{-tB^*} Q_\infty^{-1} v = -Q_\infty B^* Q_\infty^{-1} D_t\, v.$$

Writing $w = Q_\infty^{-1} D_t v$, we get

$$\frac{\partial}{\partial t} R(D_t v) = -\langle B^* w,\ Q_\infty w\rangle.$$

Now

$$\frac{\partial}{\partial t} R(D_t v) = -\langle B^* w,\ Q_\infty w \rangle$$
$$= -\int_0^\infty \left\langle B^* w,\ e^{sB}\, Q\, e^{sB^*}\, w \right\rangle\, ds$$

Now

$$\begin{aligned}
\frac{\partial}{\partial t} R(D_t v) &= -\langle B^* w,\ Q_\infty w\rangle \\
&= -\int_0^\infty \left\langle B^* w,\ e^{sB}\, Q\, e^{sB^*} w \right\rangle\, ds \\
&= -\int_0^\infty \left\langle Q^{1/2} e^{sB^*}\, B^* w,\ Q^{1/2}\, e^{sB^*} w \right\rangle\, ds \\
&= -\frac{1}{2} \int_0^\infty \frac{d}{ds} \left\langle Q^{1/2} e^{sB^*}\, w,\ Q^{1/2}\, e^{sB^*} w \right\rangle\, ds
\end{aligned}$$

Now

$$\begin{aligned}\frac{\partial}{\partial t} R(D_t v) &= -\langle B^* w,\ Q_\infty w\rangle \\ &= -\int_0^\infty \left\langle B^* w,\ e^{sB}\, Q\, e^{sB^*} w \right\rangle\, ds \\ &= -\int_0^\infty \left\langle Q^{1/2} e^{sB^*}\, B^* w,\ Q^{1/2}\, e^{sB^*} w \right\rangle\, ds \\ &= -\frac{1}{2} \int_0^\infty \frac{d}{ds} \left\langle Q^{1/2} e^{sB^*}\, w,\ Q^{1/2}\, e^{sB^*} w \right\rangle\, ds\end{aligned}$$

which equals

$$\frac{1}{2}\, \left| Q^{1/2}\, w \right|^2 = \frac{1}{2}\, \left| Q^{1/2}\, Q_\infty^{-1}\, D_t v \right|^2 \sim |D_t v|^2 > 0.$$

To prove the main theorem, we must estimate the level sets of

$$\sup_{t>0} \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,,$$

To prove the main theorem, we must estimate the level sets of

$$\sup_{t>0} \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,,$$

where $f \geq 0$ is normalized in $L^1(\gamma_\infty)$.

To prove the main theorem, we must estimate the level sets of

$$\sup_{t>0} \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,,$$

where $f \geq 0$ is normalized in $L^1(\gamma_\infty)$.

The proof consists of three parts:

To prove the main theorem, we must estimate the level sets of

$$\sup_{t>0} \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,,$$

where $f \geq 0$ is normalized in $L^1(\gamma_\infty)$.

The proof consists of three parts:

- The local part, defined by the condition $|x-u| < 1/(1+|x|)$ in the integral above.

To prove the main theorem, we must estimate the level sets of

$$\sup_{t>0} \int K_t(x, u)\, f(u)\, d\gamma_\infty(u)\,,$$

where $f \geq 0$ is normalized in $L^1(\gamma_\infty)$.

The proof consists of three parts:

- The local part, defined by the condition $|x - u| < 1/(1 + |x|)$ in the integral above.

  The global part, given by $|x - u| > 1/(1 + |x|)$, is split in two:

To prove the main theorem, we must estimate the level sets of

$$\sup_{t>0} \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,,$$

where $f \geq 0$ is normalized in $L^1(\gamma_\infty)$.

The proof consists of three parts:

- The local part, defined by the condition $|x-u| < 1/(1+|x|)$ in the integral above.

  The global part, given by $|x-u| > 1/(1+|x|)$, is split in two:
- Large t, that is, $\sup_{t>1} \dots$.

To prove the main theorem, we must estimate the level sets of

$$\sup_{t>0} \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,,$$

where $f \geq 0$ is normalized in $L^1(\gamma_\infty)$.

The proof consists of three parts:

- The local part, defined by the condition $|x-u| < 1/(1+|x|)$ in the integral above.

  The global part, given by $|x-u| > 1/(1+|x|)$, is split in two:
- Large t, that is, $\sup_{t>1} \ldots$.
- Small t, that is, $\sup_{t<1} \ldots$

To prove the main theorem, we must estimate the level sets of

$$\sup_{t>0} \int K_t(x,u)\, f(u)\, d\gamma_\infty(u)\,,$$

where $f \geq 0$ is normalized in $L^1(\gamma_\infty)$.

The proof consists of three parts:

- The local part, defined by the condition $|x-u| < 1/(1+|x|)$ in the integral above.

  The global part, given by $|x-u| > 1/(1+|x|)$, is split in two:
- Large t, that is, $\sup_{t>1} \dots$.
- Small t, that is, $\sup_{t<1} \dots$

The local part is standard.

Consider now $\sup_{t>1} \ldots,$ with the global condition.

Consider now $\sup_{t>1}\dots,$ with the global condition.

The corresponding part of the Mehler kernel is controlled by

Consider now $\sup_{t>1} \ldots,$ with the global condition.

The corresponding part of the Mehler kernel is controlled by

$$K_t^{>}(x,u) = e^{R(x)} \exp\left(- c \left| D_{-t}\, u - x \right|^2 \right) \chi_{\text{global cond.}}, \qquad t > 1.$$

Consider now $\sup_{t>1} \dots,$ with the global condition.

The corresponding part of the Mehler kernel is controlled by

$$K_t^{>}(x,u) = e^{R(x)} \exp\left(-c \left| D_{-t}\, u - x \right|^2 \right) \chi_{\text{global cond.}}, \qquad t > 1.$$

For large α, we must estimate the level set

$$\left\{ x : \sup_{t>1} \int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) > \alpha \right\}.$$

Consider now $\sup_{t>1}\dots,$ with the global condition.

The corresponding part of the Mehler kernel is controlled by

$$K_t^{>}(x,u) = e^{R(x)} \exp\left(-c\left|D_{-t}\,u - x\right|^2\right) \chi_{\text{global cond.}}, \qquad t > 1.$$

For large α, we must estimate the level set

$$\left\{x : \sup_{t>1} \int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) > \alpha\right\}.$$

If x is small in the sense that $R(x) < \log\alpha$, then $K_t^{>}(x,u) < \alpha$,

Consider now $\sup_{t>1} \dots,$ with the global condition.

The corresponding part of the Mehler kernel is controlled by

$$K_t^{>}(x,u) = e^{R(x)} \exp\left(-c\left|D_{-t}\,u - x\right|^2\right) \chi_{\text{global cond.}}, \qquad t > 1.$$

For large $\alpha,$ we must estimate the level set

$$\left\{ x : \sup_{t>1} \int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) > \alpha \right\}.$$

If x is small in the sense that $R(x) < \log\alpha,$ then $K_t^{>}(x,u) < \alpha,$
so we need only consider x with $R(x) > \log\alpha$.

Consider now $\sup_{t>1} \dots,$ with the global condition.

The corresponding part of the Mehler kernel is controlled by

$$K_t^{>}(x,u) = e^{R(x)} \exp\left(-c\left|D_{-t}\, u - x\right|^2\right) \chi_{\text{global cond.}}, \qquad t > 1.$$

For large α, we must estimate the level set

$$\left\{ x : \sup_{t>1} \int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) > \alpha \right\}.$$

If x is small in the sense that $R(x) < \log\alpha$, then $K_t^{>}(x,u) < \alpha$, so we need only consider x with $R(x) > \log\alpha$.

We use our polar coordinates with $\beta = \log\alpha$

Consider now $\sup_{t>1} \dots,$ with the global condition.

The corresponding part of the Mehler kernel is controlled by

$$K_t^{>}(x,u) = e^{R(x)} \exp\left(-c \left| D_{-t}\, u - x \right|^2 \right) \chi_{\text{global cond.}}, \qquad t > 1.$$

For large α, we must estimate the level set

$$\left\{ x : \sup_{t>1} \int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) > \alpha \right\}.$$

If x is small in the sense that $R(x) < \log \alpha$, then $K_t^{>}(x,u) < \alpha$, so we need only consider x with $R(x) > \log \alpha$.

We use our polar coordinates with $\beta = \log \alpha$ and write

$$x = D_s \tilde{x}, \qquad u = D_{s'} \tilde{u},$$

where $\tilde{x}$ and $\tilde{u}$ are on the ellipsoid $E_{\log \alpha}$.

Consider now $\sup_{t>1} \dots,$ with the global condition.

The corresponding part of the Mehler kernel is controlled by

$$K_t^{>}(x,u) = e^{R(x)} \exp\big(- c\left|D_{-t}\, u - x\right|^2\big)\, \chi_{\text{global cond.}}, \qquad t > 1.$$

For large $\alpha,$ we must estimate the level set

$$\left\{ x : \sup_{t>1} \int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) > \alpha \right\}.$$

If x is small in the sense that $R(x) < \log\alpha,$ then $K_t^{>}(x,u) < \alpha,$ so we need only consider x with $R(x) > \log\alpha$.

We use our polar coordinates with $\beta = \log\alpha$ and write

$$x = D_s \tilde{x}, \qquad u = D_{s'} \tilde{u},$$

where $\tilde{x}$ and $\tilde{u}$ are on the ellipsoid $E_{\log\alpha}$.

It can be shown geometrically that then $|D_{-t}u - x| \gtrsim |\tilde{x} - \tilde{u}|$.

Thus

$$\int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) \lesssim e^{R(D_s\tilde{x})} \int \exp\big(-c\,\big|\tilde{x}-\tilde{u}\big|^2\big)\, f(u)\, d\gamma_\infty(u).$$

Thus

$$\int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) \lesssim e^{R(D_s \tilde{x})} \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u).$$

Here the right-hand side contains no t,
so we have an estimate of $\sup_{t>1} \dots$.

Thus

$$\int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) \lesssim e^{R(D_s\tilde{x})} \int \exp\big(-c\,\big|\tilde{x}-\tilde{u}\big|^2\big)\, f(u)\, d\gamma_\infty(u).$$

Here the right-hand side contains no t,
so we have an estimate of $\sup_{t>1} \dots$.
Also, the right-hand side is increasing in s.

Thus

$$\int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) \lesssim e^{R(D_s\tilde{x})} \int \exp\big(- c\,\big|\tilde{x} - \tilde{u}\big|^2\big)\, f(u)\, d\gamma_\infty(u).$$

Here the right-hand side contains no t,
so we have an estimate of $\sup_{t>1} \dots$.
Also, the right-hand side is increasing in s.
Thus, in polar coordinates the level set where

$$e^{R(D_s\tilde{x})} \int \exp\big(- c\,\big|\tilde{x} - \tilde{u}\big|^2\big)\, f(u)\, d\gamma_\infty(u) \ > \ \alpha \tag{1}$$

Thus

$$\int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) \lesssim e^{R(D_s\tilde{x})} \int \exp\big(-c\,|\tilde{x}-\tilde{u}|^2\big)\, f(u)\, d\gamma_\infty(u).$$

Here the right-hand side contains no t,
so we have an estimate of $\sup_{t>1}\dots$.
Also, the right-hand side is increasing in s.
Thus, in polar coordinates the level set where

$$e^{R(D_s\tilde{x})} \int \exp\big(-c\,|\tilde{x}-\tilde{u}|^2\big)\, f(u)\, d\gamma_\infty(u) \;>\; \alpha \tag{1}$$

can be described as

$$\{D_s\tilde{x} : \tilde{x} \in E_{\log\alpha},\ s > s_\alpha(\tilde{x})\},$$

for some function $\tilde{x} \mapsto s_\alpha(\tilde{x}) > 0$,

Thus

$$\int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) \lesssim e^{R(D_s\tilde{x})} \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u).$$

Here the right-hand side contains no t,
so we have an estimate of $\sup_{t>1} \dots$.
Also, the right-hand side is increasing in s.
Thus, in polar coordinates the level set where

$$e^{R(D_s\tilde{x})} \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u) > \alpha \tag{1}$$

can be described as

$$\left\{D_s\tilde{x} : \tilde{x} \in E_{\log\alpha},\ s > s_\alpha(\tilde{x})\right\},$$

for some function $\tilde{x} \mapsto s_\alpha(\tilde{x}) > 0$, with equality in (1) for $s = s_\alpha(\tilde{x})$.

Thus

$$\int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) \lesssim e^{R(D_s\tilde{x})} \int \exp\big(-c\,|\tilde{x}-\tilde{u}|^2\big)\, f(u)\, d\gamma_\infty(u).$$

Here the right-hand side contains no t,
so we have an estimate of $\sup_{t>1} \dots$.
Also, the right-hand side is increasing in s.
Thus, in polar coordinates the level set where

$$e^{R(D_s\tilde{x})} \int \exp\big(-c\,|\tilde{x}-\tilde{u}|^2\big)\, f(u)\, d\gamma_\infty(u) \;>\; \alpha \tag{1}$$

can be described as

$$\{D_s\tilde{x} : \tilde{x} \in E_{\log\alpha},\ s > s_\alpha(\tilde{x})\},$$

for some function $\tilde{x} \mapsto s_\alpha(\tilde{x}) > 0$, with equality in (1) for $s = s_\alpha(\tilde{x})$.
The γ_∞ measure of this set can be computed in polar coordinates

Thus

$$\int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) \lesssim e^{R(D_s\tilde{x})} \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u).$$

Here the right-hand side contains no t,
so we have an estimate of $\sup_{t>1} \dots$.
Also, the right-hand side is increasing in s.
Thus, in polar coordinates the level set where

$$e^{R(D_s\tilde{x})} \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u) \ > \ \alpha \tag{1}$$

can be described as

$$\left\{D_s\tilde{x} : \tilde{x} \in E_{\log\alpha},\ s > s_\alpha(\tilde{x})\right\},$$

for some function $\tilde{x} \mapsto s_\alpha(\tilde{x}) > 0$, with equality in (1) for $s = s_\alpha(\tilde{x})$.

The γ_∞ measure of this set can be computed in polar coordinates and is no larger than

$$\int_{E_{\log \alpha}} \int_{s_\alpha(\tilde{x})}^{\infty} |D_s \tilde{x}| \, e^{-R(D_s \tilde{x})} \, ds \, dS(\tilde{x}).$$

Here dS is the area measure of $E_{\log \alpha}$.

$$\int_{E_{\log \alpha}} \int_{s_\alpha(\tilde{x})}^{\infty} |D_s \tilde{x}| \, e^{-R(D_s \tilde{x})} \, ds \, dS(\tilde{x}).$$

Here dS is the area measure of $E_{\log \alpha}$.

We saw that the s derivative of $R(D_s \tilde{x})$ is as large as $|D_s \tilde{x}|^2$.

$$\int_{E_{\log \alpha}} \int_{s_\alpha(\tilde{x})}^{\infty} |D_s \tilde{x}| \, e^{-R(D_s \tilde{x})} \, ds \, dS(\tilde{x}).$$

Here dS is the area measure of $E_{\log \alpha}$.

We saw that the s derivative of $R(D_s \tilde{x})$ is as large as $|D_s \tilde{x}|^2$.

Therefore, only the very first part of the integral in s will matter,

$$\int_{E_{\log\alpha}} \int_{s_\alpha(\tilde{x})}^{\infty} |D_s \tilde{x}|\, e^{-R(D_s\tilde{x})}\, ds\, dS(\tilde{x}).$$

Here dS is the area measure of $E_{\log\alpha}$.

We saw that the s derivative of $R(D_s\tilde{x})$ is as large as $|D_s\tilde{x}|^2$.

Therefore, only the very first part of the integral in s will matter, and the double integral is no larger than

$$\frac{1}{\sqrt{\log\alpha}} \int_{E_{\log\alpha}} e^{-R(D_{s_\alpha(\tilde{x})}\tilde{x})}\, dS(\tilde{x}) \tag{2}$$

Using the case of equality in (1), when $s = s_\alpha(\tilde{x})$,

$$\int_{E_{\log\alpha}} \int_{s_\alpha(\tilde{x})}^{\infty} |D_s\tilde{x}|\, e^{-R(D_s\tilde{x})}\, ds\, dS(\tilde{x}).$$

Here dS is the area measure of $E_{\log\alpha}$.

We saw that the s derivative of $R(D_s\tilde{x})$ is as large as $|D_s\tilde{x}|^2$.

Therefore, only the very first part of the integral in s will matter, and the double integral is no larger than

$$\frac{1}{\sqrt{\log\alpha}} \int_{E_{\log\alpha}} e^{-R(D_{s_\alpha(\tilde{x})}\tilde{x})}\, dS(\tilde{x}) \tag{2}$$

Using the case of equality in (1), when $s = s_\alpha(\tilde{x})$,

recall : $\quad e^{R(D_s\tilde{x})} \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u) \;>\; \alpha, \quad (1)$

$$\int_{E_{\log\alpha}} \int_{s_\alpha(\tilde{x})}^{\infty} |D_s\tilde{x}|\, e^{-R(D_s\tilde{x})}\, ds\, dS(\tilde{x}).$$

Here dS is the area measure of $E_{\log\alpha}$.

We saw that the s derivative of $R(D_s\tilde{x})$ is as large as $|D_s\tilde{x}|^2$.

Therefore, only the very first part of the integral in s will matter, and the double integral is no larger than

$$\frac{1}{\sqrt{\log\alpha}} \int_{E_{\log\alpha}} e^{-R(D_{s_\alpha(\tilde{x})}\tilde{x})}\, dS(\tilde{x}) \tag{2}$$

Using the case of equality in (1), when $s = s_\alpha(\tilde{x})$,

$$\textbf{recall}: \quad e^{R(D_s\tilde{x})} \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u) > \alpha, \tag{1}$$

we see that (2) equals

$$\frac{1}{\alpha\sqrt{\log\alpha}} \int_{E_{\log\alpha}} \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u)\, dS(\tilde{x})$$

$$\frac{1}{\alpha\sqrt{\log\alpha}} \int_{E_{\log\alpha}} \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u)\, dS(\tilde{x})$$

$$= \frac{1}{\alpha\sqrt{\log\alpha}} \int f(u) \int_{E_{\log\alpha}} \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) dS(\tilde{x})\, d\gamma_\infty(u)$$

$$\lesssim \frac{1}{\alpha\sqrt{\log\alpha}} \int f(u)\, d\gamma_\infty(u).$$

$$\frac{1}{\alpha\sqrt{\log\alpha}} \int_{E_{\log\alpha}} \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u)\, dS(\tilde{x})$$

$$= \frac{1}{\alpha\sqrt{\log\alpha}} \int f(u) \int_{E_{\log\alpha}} \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) dS(\tilde{x})\, d\gamma_\infty(u)$$

$$\lesssim \frac{1}{\alpha\sqrt{\log\alpha}} \int f(u)\, d\gamma_\infty(u).$$

Thus we get an estimate

$$\gamma_\infty\left\{x : \sup_{t>1} \int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) > \alpha\right\} \lesssim \frac{1}{\alpha\sqrt{\log\alpha}} \|f\|_{L^1(\gamma_\infty)}.$$

$$\frac{1}{\alpha\sqrt{\log\alpha}}\int_{E_{\log\alpha}}\int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\,d\gamma_\infty(u)\,dS(\tilde{x})$$

$$=\frac{1}{\alpha\sqrt{\log\alpha}}\int f(u)\int_{E_{\log\alpha}} \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) dS(\tilde{x})\,d\gamma_\infty(u)$$

$$\lesssim\frac{1}{\alpha\sqrt{\log\alpha}}\int f(u)\,d\gamma_\infty(u).$$

Thus we get an estimate

$$\gamma_\infty\left\{x:\sup_{t>1}\int K_t^{>}(x,u)\,f(u)\,d\gamma_\infty(u)>\alpha\right\}\lesssim\frac{1}{\alpha\sqrt{\log\alpha}}\|f\|_{L^1(\gamma_\infty)}.$$

Here the square root factor means a slight improvement of the ordinary weak type (1,1) estimate.

$$\frac{1}{\alpha\sqrt{\log\alpha}} \int_{E_{\log\alpha}} \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u)\, dS(\tilde{x})$$

$$= \frac{1}{\alpha\sqrt{\log\alpha}} \int f(u) \int_{E_{\log\alpha}} \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) dS(\tilde{x})\, d\gamma_\infty(u)$$

$$\lesssim \frac{1}{\alpha\sqrt{\log\alpha}} \int f(u)\, d\gamma_\infty(u).$$

Thus we get an estimate

$$\gamma_\infty\left\{x : \sup_{t>1} \int K_t^{>}(x,u)\, f(u)\, d\gamma_\infty(u) > \alpha\right\} \lesssim \frac{1}{\alpha\sqrt{\log\alpha}} \|f\|_{L^1(\gamma_\infty)}.$$

Here the square root factor means a slight improvement of the ordinary weak type (1,1) estimate. This factor is sharp.

Similar improvements were proved for the standard Ornstein-Uhlenbeck semigroup by several people, but only for each fixed t.

Similar improvements were proved for the standard Ornstein-Uhlenbeck semigroup by several people, but only for each fixed t.

In particular, Lehec found the $\sqrt{\log \alpha}$ factor.

Similar improvements were proved for the standard Ornstein-Uhlenbeck semigroup by several people, but only for each fixed t.

In particular, Lehec found the $\sqrt{\log \alpha}$ factor.

This was inspired by a conjecture due to Talagrand, in another setting.

Similar improvements were proved for the standard Ornstein-Uhlenbeck semigroup by several people, but only for each fixed t.

In particular, Lehec found the $\sqrt{\log \alpha}$ factor.

This was inspired by a conjecture due to Talagrand, in another setting.

The case $t > 1$ is done.

THAT'S ALL

THANK YOU

We now move to the case of **small t** and thus $\sup_{t<1} \ldots$.

We now move to the case of **small t** and thus $\sup_{t<1} \ldots$.

The corresponding kernel is no larger than

We now move to the case of **small t** and thus $\sup_{t<1} \ldots$.

The corresponding kernel is no larger than

$$K_t^{<}(x,u) = \frac{\exp(R(x))}{t^{n/2}} \; exp\left(-c\,\frac{|\,u - D_t x|^2}{t}\right) \chi_{\text{global cond.}}, \qquad t < 1,$$

We now move to the case of **small t** and thus $\sup_{t<1} \dots$.

The corresponding kernel is no larger than

$$K_t^<(x,u) = \frac{\exp(R(x))}{t^{n/2}}\ \ exp\left(-c\,\frac{|\,u - D_t x|^2}{t}\right)\ \chi_{\text{global cond.}}, \qquad t<1,$$

and we consider the level sets

$$\sup_{t<1} \int K_t^<(x,u)\, f(u)\, d\gamma_\infty(u)\ >\ \alpha, \qquad \alpha > 0.$$

We now move to the case of **small t** and thus $\sup_{t<1} \ldots$.

The corresponding kernel is no larger than

$$K_t^{<}(x,u) = \frac{\exp(R(x))}{t^{n/2}} \; exp\left(-c\,\frac{|\,u - D_t x|^2}{t}\right) \chi_{\text{global cond.}}, \qquad t < 1,$$

and we consider the level sets

$$\sup_{t<1} \int K_t^{<}(x,u)\, f(u)\, d\gamma_\infty(u) \; > \; \alpha, \qquad \alpha > 0.$$

Here the important part is that where the second exponential factor in $K_t^{<}$ is not small,

We now move to the case of **small t** and thus $\sup_{t<1} \ldots$.

The corresponding kernel is no larger than

$$K_t^<(x,u) = \frac{\exp(R(x))}{t^{n/2}} \; exp\left(-c\,\frac{\left|\,u - D_t x\right|^2}{t}\right) \chi_{\text{global cond.}}, \qquad t < 1,$$

and we consider the level sets

$$\sup_{t<1} \int K_t^<(x,u)\, f(u)\, d\gamma_\infty(u) \; > \; \alpha, \qquad \alpha > 0.$$

Here the important part is that where the second exponential factor in $K_t^<$ is not small, which means $\left|\,u - D_t x\right|^2 \lesssim t$.

We now move to the case of **small t** and thus $\sup_{t<1} \ldots$.

The corresponding kernel is no larger than

$$K_t^<(x,u) = \frac{\exp(R(x))}{t^{n/2}}\ exp\left(-c\,\frac{\left|\,u - D_t x\right|^2}{t}\right)\ \chi_{\text{global cond.}}, \qquad t<1,$$

and we consider the level sets

$$\sup_{t<1} \int K_t^<(x,u)\, f(u)\, d\gamma_\infty(u)\ >\ \alpha, \qquad \alpha > 0.$$

Here the important part is that where the second exponential factor in $K_t^<$ is not small, which means $\left|\,u - D_t x\right|^2 \lesssim t$.

Thus we restrict the integration to u in the ball

$$B_t(x) = B\left(D_t x, \sqrt{t}\right).$$

We now move to the case of **small t** and thus $\sup_{t<1} \dots$.

The corresponding kernel is no larger than

$$K_t^<(x,u) = \frac{\exp(R(x))}{t^{n/2}} \; exp\left(-c\,\frac{\left|\,u - D_t x\right|^2}{t}\right)\chi_{\text{global cond.}}, \qquad t<1,$$

and we consider the level sets

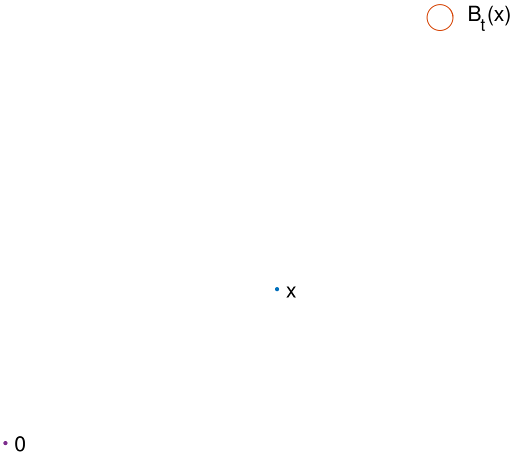
$$\sup_{t<1} \int K_t^<(x,u)\, f(u)\, d\gamma_\infty(u) \;>\; \alpha, \qquad \alpha > 0.$$

Here the important part is that where the second exponential factor in $K_t^<$ is not small, which means $\left|\,u - D_t x\right|^2 \lesssim t$.

Thus we restrict the integration to u in the ball

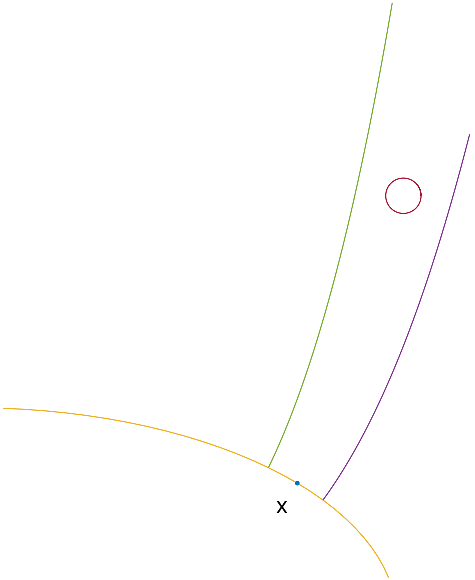
$$B_t(x) = B\left(D_t x, \sqrt{t}\right).$$

This may look as follows:


$B_t(x)$
x
0

We now introduce tube-like sets $Z(x)$, looking as follows.

The tube Z(x)



The idea is to construct a sequence of points $(x_j)_j$ such that

The idea is to construct a sequence of points $(x_j)_j$ such that

- The tubes $Z(x_j)$ together cover the level set.

The idea is to construct a sequence of points $(x_j)_j$ such that

- The tubes $Z(x_j)$ together cover the level set.
- The measures of these tubes satisfy

$$\gamma_\infty\left(Z(x_j)\right) \lesssim \int_{B_{t_j}(x_j)} f(u)\, d\gamma_\infty(u).$$

The idea is to construct a sequence of points $(x_j)_j$ such that

- The tubes $Z(x_j)$ together cover the level set.
- The measures of these tubes satisfy

$$\gamma_\infty\left(Z(x_j)\right) \lesssim \int_{B_{t_j}(x_j)} f(u)\, d\gamma_\infty(u).$$

for some numbers $t_j \in (0, 1)$.

The idea is to construct a sequence of points $(x_j)_j$ such that

- The tubes $Z(x_j)$ together cover the level set.
- The measures of these tubes satisfy

$$\gamma_\infty\left(Z(x_j)\right) \lesssim \int_{B_{t_j}(x_j)} f(u)\, d\gamma_\infty(u).$$

  for some numbers $t_j \in (0,1)$.
- The balls $B_{t_j}(x_j)$ are pairwise disjoint.

The idea is to construct a sequence of points $(x_j)_j$ such that

- The tubes $Z(x_j)$ together cover the level set.
- The measures of these tubes satisfy

$$\gamma_\infty\left(Z(x_j)\right) \lesssim \int_{B_{t_j}(x_j)} f(u)\, d\gamma_\infty(u).$$

  for some numbers $t_j \in (0,1)$.
- The balls $B_{t_j}(x_j)$ are pairwise disjoint.

Then the measure of the level set will be at most

The idea is to construct a sequence of points $(x_j)_j$ such that

- The tubes $Z(x_j)$ together cover the level set.
- The measures of these tubes satisfy

$$\gamma_\infty\left(Z(x_j)\right) \lesssim \int_{B_{t_j}(x_j)} f(u)\, d\gamma_\infty(u).$$

  for some numbers $t_j \in (0,1)$.
- The balls $B_{t_j}(x_j)$ are pairwise disjoint.

Then the measure of the level set will be at most

$$\gamma_\infty\Big(\bigcup_j Z(x_j)\Big) \lesssim \int f(u)\, d\gamma_\infty(u)$$

The idea is to construct a sequence of points $(x_j)_j$ such that

- The tubes $Z(x_j)$ together cover the level set.
- The measures of these tubes satisfy

$$\gamma_\infty \left(Z(x_j)\right) \lesssim \int_{B_{t_j}(x_j)} f(u)\, d\gamma_\infty(u).$$

  for some numbers $t_j \in (0, 1)$.
- The balls $B_{t_j}(x_j)$ are pairwise disjoint.

Then the measure of the level set will be at most

$$\gamma_\infty\Big(\bigcup_j Z(x_j)\Big) \lesssim \int f(u)\, d\gamma_\infty(u)$$

and we are done.

THAT’S ALL

THANK YOU