Regularity of solutions to $X \stackrel{d}{=} AX + B$ recursion with triangular matrices

Witold Świątkowski

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Kesten proved that under these conditions there is a unique stationary solution $\mathbf{X}$ to (1).

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Kesten's assumption

$\{\log \rho(\mathbf{M}) : \mathbf{M} \in \operatorname{supp} P_{\mathbf{A}}\}$ generates additive subgroup of $\mathbb{R}$.

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Kesten's positivity condition

There is n such that $\mathbb{P}(\mathbf{A}_0 \cdots \mathbf{A}_n > \mathbf{0}) > 0$

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There are positive constants C_k for $k = 1, \dots, d$ such that

$$\mathbb{P}(X_k > t) \sim C_k \cdot t^{-\alpha} \quad \text{as } t \to \infty$$

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The number $\alpha > 0$ (called the *tail index*) depends on the law of $\mathbf{A}$.

Simple example of a non-Kesten situation

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The stochastic recurrence equation takes the form

$$\begin{pmatrix} X_{1,n+1} \\ \vdots \\ X_{d,n+1} \end{pmatrix} = \begin{pmatrix} A_{11,n+1} & & 0 \\ & \ddots & \\ 0 & & A_{dd,n+1} \end{pmatrix} \begin{pmatrix} X_{1,n} \\ \vdots \\ X_{d,n} \end{pmatrix} + \begin{pmatrix} B_{1,n+1} \\ \vdots \\ B_{d,n+1} \end{pmatrix}$$

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Derive the tail index α_k from the law of A_{kk} for $k = 1, \ldots, d$. Then

$$\mathbb{P}(X_k > t) \sim C_k \cdot t^{-\alpha_k},$$

so the tail behaviour can be different for different coordinates.

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Then we deduce negativity of the *top Lyapunov exponent*

but we need one more crucial assumption!

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The index $\tilde{\alpha}_i$ can depend on $\alpha_i, \alpha_{i+1}, \dots, \alpha_d$.

The new index $\tilde{\alpha}_i$

Example: $d = 2$ (see a paper by E. Damek, M. Matsui and W.Ś.)

Consider the matrix

$$\mathbf{A}_n = \begin{pmatrix} A_{11,n} & A_{12,n} \\ 0 & A_{22,n} \end{pmatrix}$$

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Definition of $\tilde{\alpha}_i$

$$\tilde{\alpha}_i = \alpha_i \wedge \min\{\alpha_j : i \text{ depends on } j\}$$

Assumptions and the theorem

Assume that the conditions of Theorem 1 are satisfied, but one assumption is weakened:

$$\alpha_i \neq \alpha_j \text{ for } i \neq j \quad \longmapsto \quad A_{ii} \not\equiv A_{jj} \text{ for } i \neq j$$

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Then the following theorem holds:

Theorem 2. (W. Ś.)

Under the mentioned assumptions, there are positive L_i, M_i and ξ_i such that for t large enough

$$L_i t^{-\tilde{\alpha}_i} \leq \mathbb{P}(X_i > t) \leq M_i t^{-\tilde{\alpha}} (\log t)^{\xi(i)}$$

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What is the best upper bound we can obtain?

Optimal exponent

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Conjecture: $\xi(i) = r(i)$, evidence in 2×2 case.

The end

Thank you!

Very rough sketch of the proof

We estimate $\mathbb{P}(X_i > t)$ using the representation

$$X_{i,n} = A_{ii,n}X_{i,n-1} + \cdots + A_{dd,n}X_{d,n-1} + B_{i,n} = A_{ii,n}X_{i,n-1} + D_{i,n}$$

from which it follows that

$$X_{i,n} = \sum_{k=0}^{\infty} A_{i,n} \cdots A_{i,n-k-1} D_{i,n-k}.$$

We set $n(t) = \rho_i^{-1} \log t + D\sqrt{\log \log t}$ and divide the series into two parts:

$$Q_F(t) = X_{i,n} = \sum_{k=0}^{n(t)} A_{i,n} \cdots A_{i,n-k-1} D_{i,n-k}$$

$$Q_T(t) = X_{i,n} = \sum_{k=n(t)+1}^{\infty} A_{i,n} \cdots A_{i,n-k-1} D_{i,n-k}$$

- To estimate $Q_F(t)$ we improve techniques developed by E. Damek and J. Zienkiewicz for the case $A_{ii} = A_{jj}$.
- Rough estimation of $Q_T(t)$ follows inductively by Chebyshev's inequality.
- The proof of $\xi(i) \leq r(i) + 1$ follows by induction as well, but it is long, technical, much more delicate and the main argument is combinatorial.

$A_{ii,n} = A_{jj,n}$ a.s.

$$\mathbf{A}_n = \begin{pmatrix} A_n & D_n \\ 0 & A_n \end{pmatrix}$$

α is the tail index derived from the law of A.
E. Damek and J. Zienkiewicz proved that either

$$\mathbb{P}(X_1 > t) \sim C \cdot t^{-\alpha}(\log t)^{\alpha},$$

or

$$\mathbb{P}(X_1 > t) \sim C \cdot t^{-\alpha}(\log t)^{\alpha/2}$$

depending on the value of $\mathbb{E}\left[D \cdot A^{\alpha-1}\right]$.