Interfaces in planar Ising and Potts models

a review

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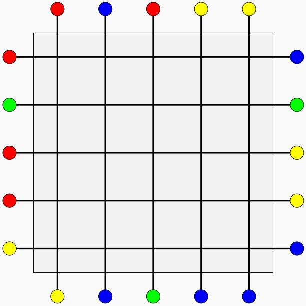
Definition of the models

- **Box:** $\Lambda \Subset \mathbb{Z}^2$
- **Boundary condition:** $\eta \in \{1, \ldots, q\}^{\partial\Lambda}$



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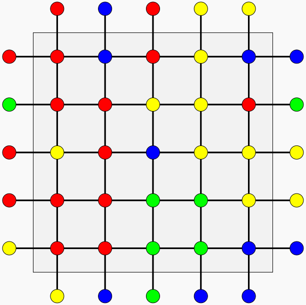
- **Configurations in Λ:**

$$\Omega_\Lambda = \{1, \ldots, q\}^\Lambda$$

The Ising model corresponds to $q = 2$



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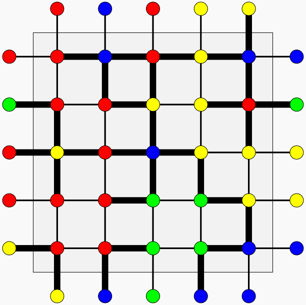
- **Energy of** $\sigma \in \Omega_\Lambda$ **with b.c.** η**:**

$$\mathcal{H}_{\Lambda;\eta}(\sigma) = \sum_{\substack{i,j\in\Lambda \\ i\sim j}} \mathbb{1}_{\{\sigma_i \neq \sigma_j\}} + \sum_{\substack{i\in\Lambda,\, j\in\partial\Lambda \\ i\sim j}} \mathbb{1}_{\{\sigma_i \neq \eta_j\}}$$



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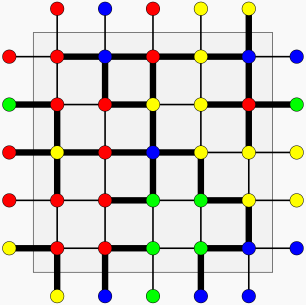


- **Gibbs measure in** Λ **with boundary condition** η**, at inverse temperature** $\beta \geq 0$**:**

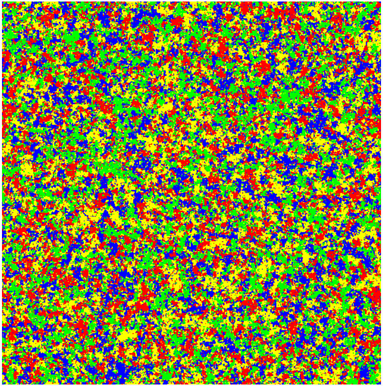
$$\mu^\eta_{\Lambda;\beta}(\sigma) = \frac{1}{\mathcal{Z}^\eta_{\Lambda;\beta}}\, e^{-\beta\mathcal{H}_{\Lambda;\eta}(\sigma)}$$

  where $\mathcal{Z}^\eta_{\Lambda;\beta} = \sum_{\sigma\in\Omega_\Lambda} e^{-\beta\mathcal{H}_{\Lambda;\eta}(\sigma)}$ is the **partition function**

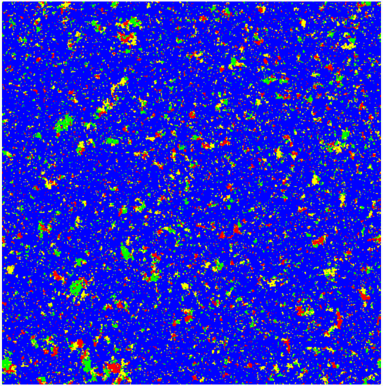
- **Measures with constant b.c.:** $\mu^k_{\Lambda;\beta} = \mu^{\eta_k}_{\Lambda;\beta}$, where $1 \leq k \leq q$ and $\eta_k \equiv k$

Let $\beta_c = \log(1 + \sqrt{q})$ be the **critical inverse temperature**.

Typical configurations under $\mu^1_{\Lambda;\beta}$:



$\beta < \beta_c$



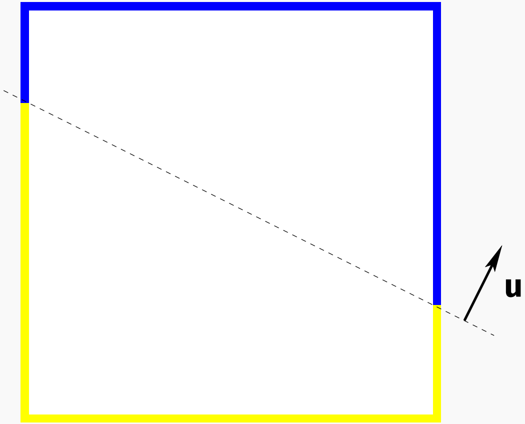
$\beta > \beta_c$

In the sequel: **we always assume that $\beta > \beta_c$.**

Definition of the interface

Properties of the interface: definition

Consider the Potts model in a box with **Dobrushin boundary condition**:



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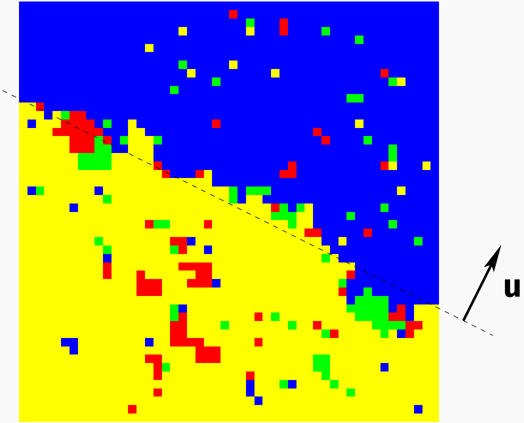

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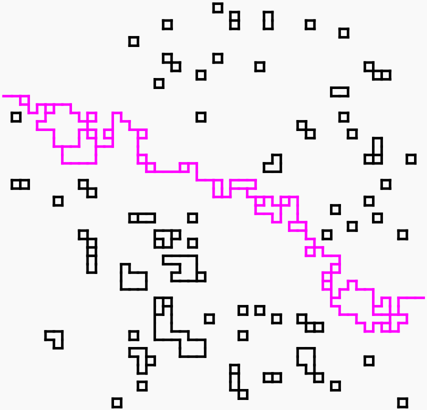


We are interested in the behavior of the **interface** (the set of purple edges).

Properties of the interface: some milestones

Profile of expected magnetization

- **Abraham, Reed 1976: $\mathbf{u} = \mathbf{e}_2$,** Ising, $\beta > \beta_c$
- **Abraham, Upton 1988:** arbitrary **u**, Ising, $\beta > \beta_c$

Microscopic structure

- **Bricmont, Lebowitz, Pfister 1981: $\mathbf{u} = \mathbf{e}_2$,** Ising, $\beta \gg 1$
- **Campanino, Ioffe, V. 2003:** arbitrary **u**, Ising, $\beta > \beta_c$
- **Campanino, Ioffe, V. 2008:** arbitrary **u**, Ising, $\beta > \beta_c$

Fluctuations

- **Gallavotti 1972:** order $n^{1/2}$, $\mathbf{u} = \mathbf{e}_2$, Ising, $\beta \gg 1$
- **Higuchi 1979:** invariance principle, $\mathbf{u} = \mathbf{e}_2$, Ising, $\beta \gg 1$
- **Greenberg, Ioffe 2005:** invariance principle, arbitrary **u**, Ising, $\beta > \beta_c$
- **Campanino, Ioffe, V. 2008:** invariance principle, arbitrary **u**, Potts, $\beta > \beta_c$

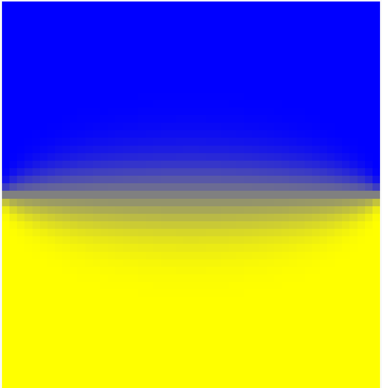
Properties of the interface: profile of expected magnetization

Roughly speaking, **explicit** computation of the profile

$$i \mapsto \mu^{\mathbf{e}_2}_{\Lambda_n;\beta}(\sigma_i = 1)$$

for the planar Ising model.

The “transition region” has width $O(\sqrt{n})$.



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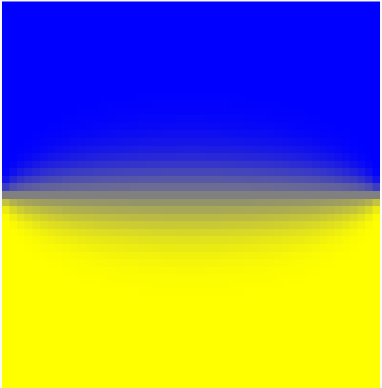

PRO

explicit expressions, including constants

CON

requires integrability
provides little understanding
no info on typical configurations
no direct access to interface

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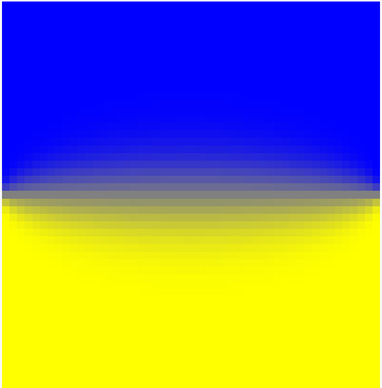
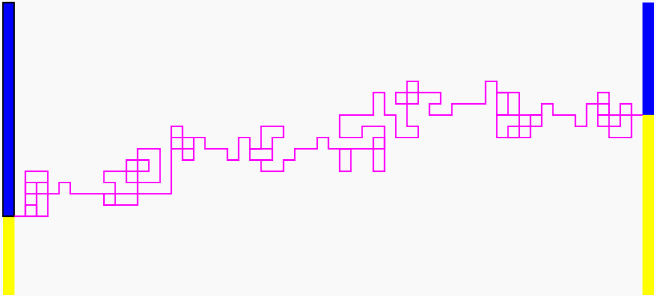


In particular, results **compatible with various scenarios**, including:

- "fat" interface, of width $\sim \sqrt{n}$
- "string-like" interface, exhibiting Gaussian fluctuations with variance $\sim n$

Properties of the interface: microscopic structure



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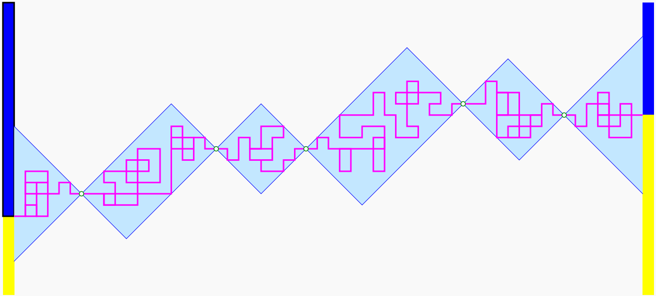

Ornstein–Zernike theory (Campanino, Ioffe, V. 2003, 2008 and Ott, V., 2018):

- concatenation of independent microscopic pieces (exponential tails)

  $\rightsquigarrow$ interface has **bounded average width**

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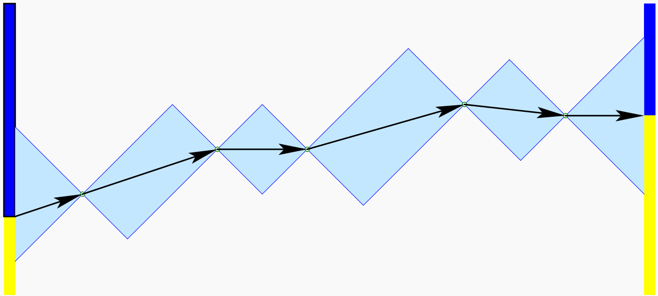

Ornstein–Zernike theory (Campanino, Ioffe, V. 2003, 2008 and Ott, V., 2018):

- concatenation of independent microscopic pieces (exponential tails)
  $\rightsquigarrow$ interface has **bounded average width**
- strong form of coupling with a **directed random walk**
  $\rightsquigarrow$ enables detailed analysis of fluctuations

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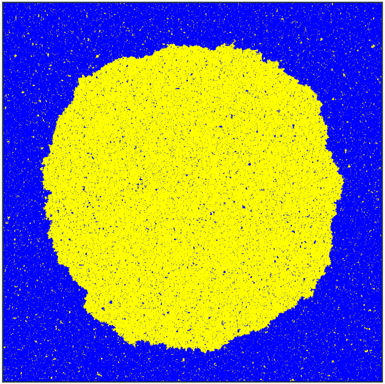
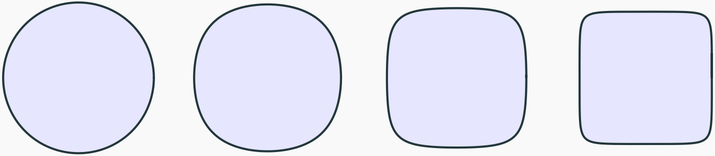


The emergence of the Wulff shape in the continuum limit of 2d systems with fixed "magnetization" has been understood since the 1990s (Dobrushin, Kotecký, Shlosman, Pfister, Ioffe, Schonmann, V., ...).

Parenthesis: regularity properties of the Wulff shape

Theorem (Campanino, Ioffe, V. 2003, 2008)

*The boundary of the Wulff shape of the Potts model on $\mathbb{Z}^2$ at any $\beta > \beta_c$ is **analytic** and **strictly convex**, with an **everywhere positive curvature**.*



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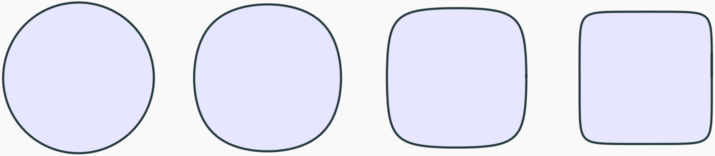


In particular, the Wulff shape of the 2d Potts model has no facet, at any positive temperature $\rightsquigarrow$ **no roughening transition** in the 2*d* Potts model.

Back to interface fluctuations

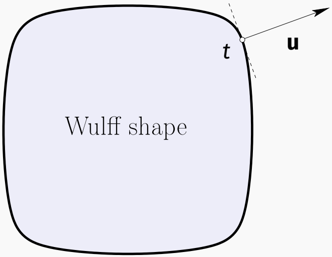
Properties of the interface: fluctuations

Theorem (Greenberg, Ioffe 2005, Campanino, Ioffe, V. 2008)

Let **u** *be a unit vector in* $\mathbb{R}^2$ *and* $\beta > \beta_c$.
The interface of the 2d Potts model in the direction **u** *weakly converges, under diffusive scaling, to the distribution of*

$$\sqrt{\chi_\beta}\, B_t$$

where B_t *is the standard Brownian bridge on* $[0, 1]$ *and* χ_β *is the curvature of the Wulff shape at the unique point* t *of its boundary where the normal is* **u**.



Pinning of the interface

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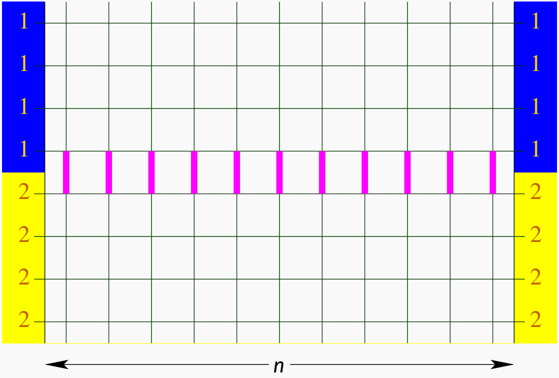

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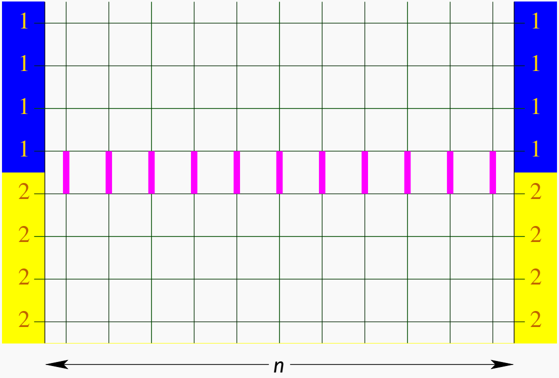


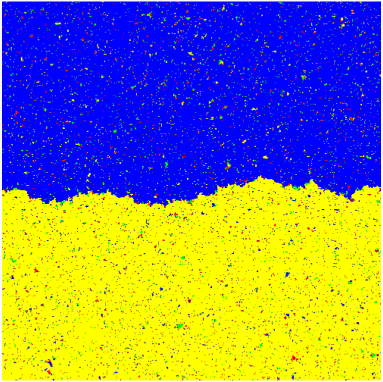
- row of modified coupling constants (purple), with value $J \geq 0$ instead of 1
- first considered by Abraham in 1981 for the 2d Ising model (exact computations)

Pinning of the interface

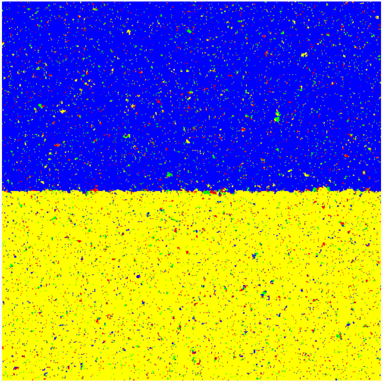
Consider the above setting for an arbitrary Potts model on $\mathbb{Z}^2$.

Theorem (Ott, V. 2018)

The interface is localized (fluctuations have bounded variance) for all $J < 1$.



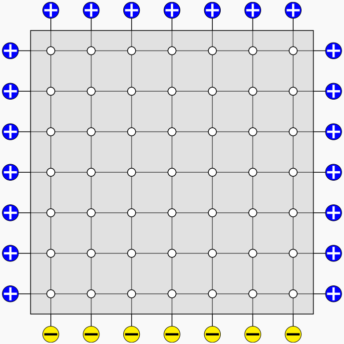
$J = 1$



$J = \frac{1}{2}$

Entropic repulsion and critical prewetting

We consider an Ising model in a square box of sidelength n, with the following boundary condition:



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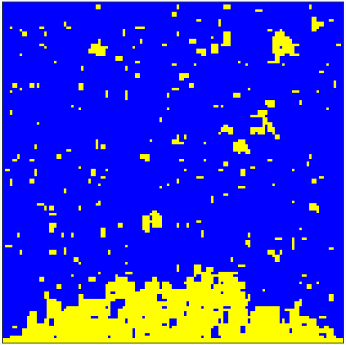

Theorem (Ioffe, Ott, V., Wachtel 2019)

The interface weakly converges, under diffusive scaling, to $\sqrt{\chi_\beta}\, \mathrm{e}_t$*, where* e_t *is the standard Brownian excursion. (This holds for general Potts models.)*

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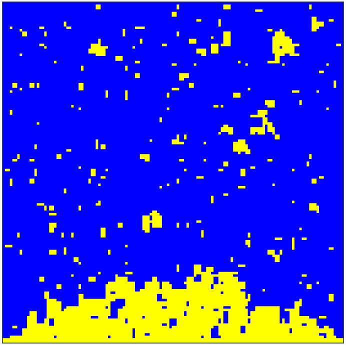

Introduce now a **magnetic field** h>0 favoring blue spins

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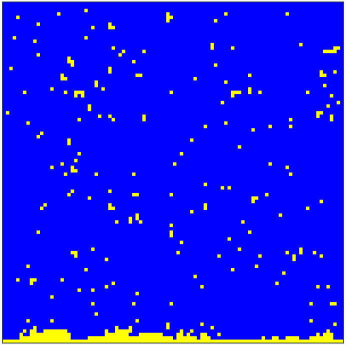
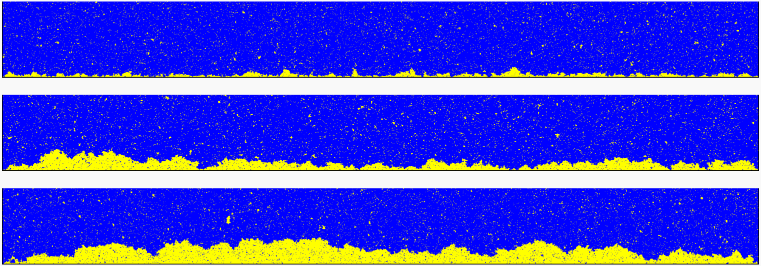


Introduce now a **magnetic field** h>0 favoring blue spins

- $\rightsquigarrow$ yellow phase becomes **thermodynamically unstable**
- $\rightsquigarrow$ layer becomes **microscopic**

Entropic repulsion & critical prewetting

However, the width of this layer increases as $h \downarrow 0$:



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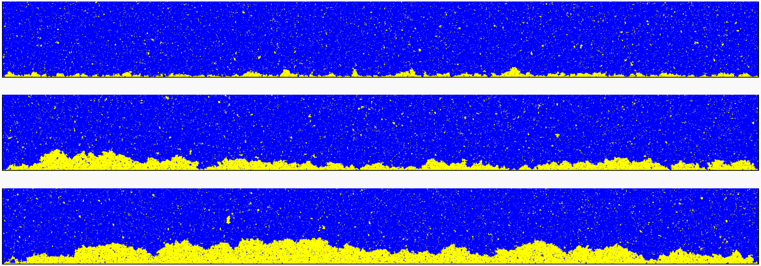

We are interested in the **scaling limit of this layer** as $n \to \infty$ and $h \downarrow 0$.

A good choice is to set

$$h = h(n) = \frac{\lambda}{n}$$

for some $\lambda > 0$.

- **Scale** the interface by $n^{-1/3}$ vertically, $n^{-2/3}$ horizontally
- Let $m_\beta^* = \langle \sigma_0 \rangle_\beta^+ =$ denote the spontaneous magnetization
- Let
$$\varphi_0(r) = \mathrm{Ai}\big((4m_\beta^*/\chi_\beta)^{1/3} r + \omega_1\big)$$
where Ai is the Airy function and ω_1 its first zero.

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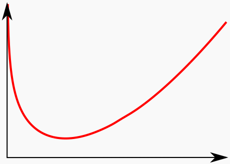
Theorem (Ioffe, Ott, Shlosman, V. 2019)

As $n \to \infty$, the scaled interface weakly converges to the diffusion on $\mathbb{L}^2(\mathbb{R}^+)$ with generator

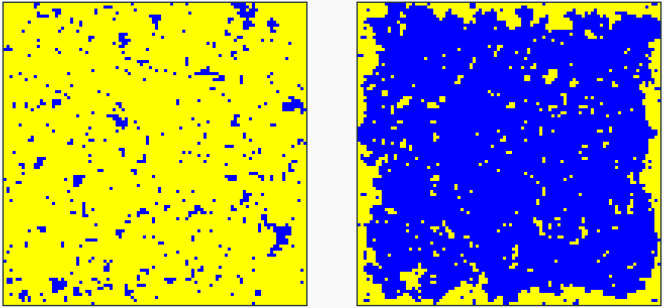
$$\tfrac{1}{2}\chi_\beta \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \chi_\beta \frac{\varphi_0'}{\varphi_0} \frac{\mathrm{d}}{\mathrm{d}r}\,.$$



Entropic repulsion & critical prewetting

Alternative geometric setting: box of sidelength n with $-$ boundary condition on all sides and a positive magnetic field $h = \lambda/n$.

It is well known [Schonmann, Shlosman 1996] that there exists $\lambda_c \in (0, \infty)$ such that:



$\lambda < \lambda_c$ $\lambda > \lambda_c$

When $\lambda > \lambda_c$, the layer of unstable phase away from the corners will have the same scaling limit as before.

Thank you for your attention!

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