Regularity of solutions to SPDEs

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Introduction

Stochastic heat equation on $\mathbb{R}^d$:

$$du + \Delta u\,\mathrm{d}t = \phi\,\mathrm{d}W, \qquad u(0) = 0$$

- $\phi : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ measurable and adapted
- W standard Brownian motion
- Reformulation $X_0 = L^p(\mathbb{R}^d)$:

$$U : \mathbb{R}_+ \times \Omega \to X_0, \qquad U(t,\omega)(x) = u(t,\omega,x)$$

$$dU + AU\,\mathrm{d}t = \phi\,\mathrm{d}W, \qquad U(0) = 0$$

  $\phi \in L^p(\mathbb{R}_+ \times \Omega; X_0)$ implies space/time regularity of U
- Stochastic convolution with heat semigroup e^{-tA}:

$$U(t) = \int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}W(s)$$

- Optimal time or space regularity of U ?

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Overview

Stochastic integration in Banach spaces

- UMD Banach space: Bourgain–Burkholder 1980-1986
  E.g. $X = L^q$ with $q \in (1, \infty)$
- X type 2: $\mathbb{E}\Big\| \sum_n r_n x_n\Big\|^2 \leq C \sum_n \|x_n\|^2$
  E.g. $X = L^q$ with $q \in [2, \infty)$
- Seidler 2010 , X UMD and type 2

$$\Big(\mathbb{E} \sup_t \|(\phi \cdot W)\|_X^p\Big)^{1/p} \leq C_X \sqrt{p} \Big(\mathbb{E}\|\phi\|^p_{L^2(\mathbb{R}_+;X)}\Big)^{1/p}$$

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$$c_{p,q}(\mathbb{E}\|\phi\|^p)^{1/p} \leq \Big(\mathbb{E} \sup_t \|(\phi \cdot W)\|_X^p\Big)^{1/p} \leq C_q p (\mathbb{E}\|\phi\|^p)^{1/p}$$

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$W : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ - standard Brownian motion

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$$\|f\|_{B^{1/2}_{p,\infty}(0,1)} = \sup_{h\in(0,1)} h^{-1/2} \|f(\cdot + h) - f\|_{L^p(0,1-h)} \quad \text{(Besov space)}$$

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What about stochastic integrals in infinite dimensions?

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- $N_p(t) = (t\sqrt{\log(t+1)})^p$
- X has UMD and type 2

Theorem (Ondreját-V. 2018)

If ϕ is bounded a.s., then $\phi \cdot W \in B^{1/2}_{\Phi,\infty}(0,1;X)$ a.s. and

1. $\phi \mapsto \phi \cdot W$ *is continuous in probability*
2. $\|\phi \cdot W\|_{L^p(\Omega;B^{1/2}_{\Phi,\infty})} \leq C\sqrt{p}\|\phi\|_{L^\infty(\Omega;L^\infty)}$ *for all* $p \in [1,\infty)$
3. $\|\phi \cdot W\|_{L^p(\Omega;B^{1/2}_{\Phi,\infty})} \leq C\sqrt{p}\|\phi\|_{L^{N_p}(\Omega;L^\infty)}$ *for all* $p \in [1,\infty)$

- L^p-norms of stochastic integrals Seidler 2010
- Discretization argument inspired by Hytönen–V. 2008
- Lenglart type extrapolation to get self-improvement (2) $\Rightarrow$ (3)

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- L^p-convolution estimate in Besov spaces via:

$$U(t) = -A\int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}s + \phi \cdot W$$

Maximal L^p-regularity in space I

Maximal regularity plays an important role in evolution equations
Monographs in deterministic case

- Amann 1995 volume 1
- Lunardi 1995
- Denk–Hieber–Prüss 2003
- Kunstmann-Weis 2004
- Prüss–Simonett book 2016
- Hytönen–van Neerven–V.–Weis: Analysis in Banach spaces 3, 2020

Contributors on maximal regularity for SPDEs:

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Maximal L^p-regularity in space II

- $X_1 \overset{d}{\hookrightarrow} X_0$ Banach spaces with UMD and type 2
- $X_{\frac{1}{2}} = [X_0, X_1]_{\frac{1}{2}}$ complex interpolation space
- $-A$ generator of C_0-semigroup on X_0 and $D(A) = X_1$

We consider again

$$dU + AU\,\mathrm{d}t = \phi\,\mathrm{d}W, \quad U(0) = 0 \tag{1}$$

Solution $U(t) = \int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}W(s)$

Definition

For $p \in [2,\infty)$ and $T \in (0,\infty]$, we say that A has stochastic maximal L^p-regularity on $(0,T)$ if for all adapted $\phi \in L^p((0,T)\times\Omega;\gamma(H,X_0))$, $U \in L^p((0,T)\times\Omega;X_{1/2})$ and $\exists C \geq 0$ independent g such that

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Kernel: $e^{-(t-s)A} \in \mathcal{L}(X_0, X_{1/2})$ singularity $(t-s)^{-1/2}$

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Theorem (Agresti–V. 2019, Lorist–V. 2019)

Stochastic maximal L^p-regularity

(i) *is independent of the dimension of the noise/dim(H)*

(ii) *implies $(e^{-tA})_{t\geq 0}$ is an analytic semigroup*

(iii) *on $(0,\infty)$ implies $(e^{-tA})_{t\geq 0}$ is exponentially stable*

(iv) *on $(0,T_0)$ implies the same on $(0,T_1)$ for any $T_1 \in (0,\infty)$*

(v) *on $(0,T)$ for $T<\infty$ and $\omega_0(A)<0$ implies the same for $T=\infty$*

(vi) *with weight t^α is independent of $\alpha \in (-1, \frac{p}{2}-1)$*

(vii) *for some $p \in [2,\infty)$ implies the same for all $p \in (2,\infty)$*

- (i)–(v) deterministic case Dore's 2000
- (vi) stochastic version of Prüss–Simonett 2004
- (vii) uses stochastic Calderón-Zygmund theory (Emiel Lorist's talk)

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(ii) *implies $(e^{-tA})_{t\geq 0}$ is an analytic semigroup*

(iii) *on $(0,\infty)$ implies $(e^{-tA})_{t\geq 0}$ is exponentially stable*

(iv) *on $(0,T_0)$ implies the same on $(0,T_1)$ for any $T_1 \in (0,\infty)$*

(v) *on $(0,T)$ for $T<\infty$ and $\omega_0(A)<0$ implies the same for $T=\infty$*

(vi) *with weight t^α is independent of $\alpha \in (-1, \frac{p}{2}-1)$*

(vii) *for some $p \in [2,\infty)$ implies the same for all $p \in (2,\infty)$*

- (i)–(v) deterministic case Dore's 2000
- (vi) stochastic version of Prüss–Simonett 2004
- (vii) uses stochastic Calderón-Zygmund theory (Emiel Lorist's talk)

Maximal L^p-regularity in space III

Theorem (Agresti–V. 2019, Lorist–V. 2019)

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Let $X_0 = L^q$ with $q \in [2, \infty)$. If A has a bounded H^∞-calculus of angle $< \pi/2$, then A has stochastic maximal L^p-regularity for all $p \in (2, \infty)$. Here $p = q = 2$ is allowed as well

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Proofs in Hilbert space, $p = 2$ case

- A has a bounded H^∞-calculus of angle $< \pi/2$ iff $D(A^{1/2}) = X_{1/2}$ iff

$$\int_{\mathbb{R}_+} \|A^{1/2} e^{tA} x\|^2 \, \mathrm{d}t \eqsim \|x\|^2$$ square function estimate

Using this estimate we obtain:

$$\begin{aligned}
\|A^{1/2} U\|^2_{L^2(\mathbb{R}_+ \times \Omega; X_0)} &= \int_0^\infty \int_0^t \|A^{1/2} e^{-(t-s)A} \phi(s)\|^2 \, \mathrm{d}s \, \mathrm{d}t \\
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Operators A depending on (t, ω)

- $A : \mathbb{R}_+ \times \Omega \to \mathcal{L}(X_1, X_0)$ adapted and measurable

Definition

For $p \in [2, \infty)$ and $T \in (0, \infty]$, we say that $A \in \mathrm{SMR}(p, T)$ if for all adapted $f \in L^p((0, T) \times \Omega; X_0)$ and $\phi \in L^p((0, T) \times \Omega; \gamma(H, X_{1/2}))$ there exist an adapted $U \in L^p((0, T) \times \Omega; X_{1/2})$ such that

$$dU + AU\,\mathrm{d}t = f\,\mathrm{d}t + \phi\,\mathrm{d}W, \quad u(0) = 0 \tag{2}$$

and there is a C independent of f and ϕ such that

$$\|U\|_{L^p((0,T)\times\Omega;X_1)} \leq C(\|f\|_{L^p((0,T)\times\Omega;X_0)} + \|\phi\|_{L^p((0,T)\times\Omega;\gamma(H,X_{1/2}))}$$

We write $A \in \mathrm{DMR}(p, T)$ if the above holds with $\phi = 0$

$A \in \mathrm{SMR}(p, T)$ implies well-posedness for semilinear equations

Reduction to t-independent setting and DMR

Theorem (Portal–V. 2019)

Let $p \in [2, \infty)$. Assume the following conditions:

1. *There exists an operator $A_0 \in \mathcal{L}(X_1, X_0)$ with $A_0 \in SMR(p, T)$*
2. *$A(\cdot, \omega) \in DMR(p, T)$ with estimates independent of $\omega \in \Omega$*

Then $A \in SMR(p, T)$

Proof.

$$dU + AU\,\mathrm{d}t = F\,\mathrm{d}t + \phi\,\mathrm{d}W$$
$$dV + A_0V\,\mathrm{d}t = F\,\mathrm{d}t + \phi\,\mathrm{d}W$$

Then V has the required regularity, because $A_0 \in \mathrm{SMR}(p, T)$
$Z = U - V$ satisfies: $Z' + AZ = A_0V - AV$. Then Z has the required regularity because $A \in \mathrm{DMR}(p, T)$ and the right measurability.
Thus $U = V + Z$ has the required regularity. □

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Then $A \in \mathcal{SMR}(p, T)$.

- Applications to SPDEs with $2m$-th order elliptic system: coefficients adapted measurable in (t, ω), VMO in space
- These results extend Krylov's theory to complex systems
- Additionally maximal estimates: $U \in L^p(\Omega; C([0, T]; (X_0, X_1)_{1-\frac{1}{p},p}))$

Continuity in time I

Pathwise continuity of $U(t) = \int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}W(s)$?

Issue: $U(t) + \int_0^t AU(s)\,\mathrm{d}s = \int_0^t \phi(s)\,\mathrm{d}W(s)$ only for regular ϕ

Theorem (Kotelenez 1984, Tubaro 1984, Ichakawa 1986)

Let X be a Hilbert space, assume $\|e^{-tA}\| \leq 1$ and $\phi \in L^2(0,T;X)$ a.s. Then U has a version with continuous paths and for all $p \in (0,\infty)$,

$$\mathbb{E}\max_{t\in[0,T]} \|U(t)\|^p \leq C_p \mathbb{E}\|\phi\|^p_{L^2(0,T;X)}$$

- Neerven–Zhu 2011 X is a 2-smooth Banach space, e.g. $L^q, q \geq 2$
  Itô's formula applied to $\|U(t)\|^2$ (not C^2 in general!!!)

Open problem: Does there always exist a continuous version ?

- Da Prato–Kwapień–Zabczyk 1987: $\phi \in L^p(0,T;X)$ for $p > 2$
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Hausenblass–Seidler 2001. Different approach. X is a Hilbert space:

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For groups the maximal estimate is very easy:

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Setting $M = \sup_{|t|\leq T}\|e^{-tA}\|$ we find

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- systems on L^q with $q \neq 2$

H^∞-calculus still ok and can be used as black box

Theorem (Seidler 2010, V.–Weis 2011)

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Stochastic Volterra equations on Hilbert spaces X:

- Peszat–Zabczyk 2014: self-adjoint positive A
- Schnaubelt–V. 2017: A bounded H^∞-calculus of angle $< \pi/2$

Proof based on positive definiteness techniques:

Theorem (Naĭmark 1943)

Let $R(t)$ be C_0, $R(0) = I$ and set $R(-t) = R(t)^*$. TFAE:

- *R dilates to a C_0-group on a larger Hilbert space*
- $\sum_{n,m}(x_m, R(t_n - t_m)x_n) \geq 0$ *(positive definite)*

This can be applied to the resolvent of the Volterra equation
The real difficulty is to prove the positive definiteness

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Summary/future work and open problems

- Time regularity of order $1/2$ holds in the scale of Besov spaces $B^{\frac{1}{2}}_{p,\infty}$, $B^{\frac{1}{2}}_{\Phi,\infty}$ both for stochastic integrals and convolutions
- The H^∞-functional calculus implies space regularity of order $\frac{1}{2}$
- Both contractivity and the H^∞-functional calculus can be used to obtain maximal estimates

Open problems:

1. Constant $p \to \sqrt{p}$ in BDG inequality for UMD spaces:

$$\Big(\mathbb{E}\sup_t \|(\phi\cdot W)\|_X^p\Big)^{1/p} \le C\sqrt{p}\Big\|\Big(\int_{\mathbb{R}_+}|\phi|^2\,\mathrm{d}t\Big)^{1/2}\Big\|_X$$

2. Characterization of stochastic maximal L^p-regularity
3. Continuity of stochastic convolution $U(t) = \int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}W(s)$ for $\phi \in L^2(\mathbb{R}_+; X)$ a.s. for Hilbert spaces X.

Summary/future work and open problems

- Time regularity of order $1/2$ holds in the scale of Besov spaces $B^{\frac{1}{2}}_{p,\infty}$, $B^{\frac{1}{2}}_{\Phi,\infty}$ both for stochastic integrals and convolutions
- The H^∞-functional calculus implies space regularity of order $\frac{1}{2}$
- Both contractivity and the H^∞-functional calculus can be used to obtain maximal estimates

Open problems:

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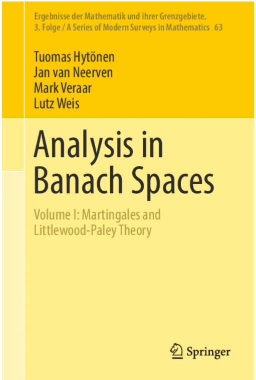

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