On the potential theory of jump processes in open sets

Zoran Vondraček

University of Zagreb, Croatia
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Joint work with P. Kim (SNU) and R. Song (UIUC)

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A typical example is a Lévy process in $\mathbb{R}^d$ where $J(x, y) = j(|x - y|)$ with j the density of the Lévy measure. The most familiar case is the isotropic α-stable process with $J(x, y) = |x - y|^{-d-\alpha}$.

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Suppose that the underlying state space E is a metric space with distance d. There are various modifications of the above kernels, usually of the following type: There exists $c \geq 1$ such that

$$c^{-1} f(d(x,y)) \leq J(x,y) \leq c f(d(x,y)), \quad x, y \in E,$$

with $f : [0,\infty) \to (0,\infty]$ bounded away from zero on compact sets.

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with $f:[0,\infty)\to(0,\infty]$ bounded away from zero on compact sets. This includes stable-like processes where $J(x,y)=c(x,y)|x-y|^{-d-\alpha}$ where $c(x,y)$ is bounded and bounded away from zero.

More general kernels of jump processes

Bogdan, Kumagai and Kwaśnicki (2015) studied jump processes in a metric space E with the kernel $J(x,y)$ satisfying the following assumption: For every $x_0 \in E$ and all radii $0 < r < R < R_0$ ($R_0 \in (0,\infty]$ is the localization radius), there exists a constant $c \geq 1$ such that for all $x \in B(x_0, r)$ and all $y \in E \setminus B(x_0, R)$

$$c^{-1} J(x_0, y) \leq J(x,y) \leq cJ(x_0, y)\,.$$

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The most common way to do that is to kill the jump process X when it exits D for the first time. The obtained process is denoted by X^D and called the killed process, or the part of X in D. Analytically, its infinitesimal generator is, roughly, the generator of X with zero exterior condition.

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Hence, the Dirichlet form of the killed process looks like

$$\begin{aligned}\mathcal{E}(u,v) &= \frac{1}{2}\iint_{D\times D}(u(x)-u(y))(v(x)-v(y))J(x,y)dydx \\ &\quad + \int_D u(x)v(x)\kappa(x)\, dx\end{aligned}$$

Censored process

Another type of processes that were studied in open subsets D are censored ones. One way to get a censored processes is to remove the killing part from the Dirichlet form of the killed process and get the form

$$\mathcal{C}(u,v)=\frac{1}{2}\iint_{D\times D}(u(x)-u(y))(v(x)-v(y))J(x,y)dydx.$$

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For example, with $J(x,y) = |x-y|^{-d-\alpha}$ one gets the censored α-stable process.
Note that the censored process is intrinsically defined in D and its not part of some larger process.

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Boundary Harnack principle

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BHP for diffusions (i.e., local operators): $X = (X_t, \mathbb{P}_x)$ a continuous strong Markov process in $\mathbb{R}^d$. The scale invariant BHP (or a local BHP) holds in an open set $D \subset \mathbb{R}^d$ if there exist $C > 0$ and $r_0 > 0$ such that for all $Q \in \partial D$, all $r \in (0, r_0)$ and all non-negative functions f and g harmonic in $B(Q, r) \cap D$ that vanish continuously on $B(Q, r) \cap \partial D$ it holds that

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It is well known that the BHP holds for Brownian motion (more generally, elliptic diffusions) in Lipschitz domains (Ancona 1978, Wu 1978, Dahlberg 1977 global BHP), NTA domains (Jerison-Kenig 1982), uniform domains (Aikawa 2001). In case of smooth boundary (say $C^{1,1}$) one gets the BHP with explicit decay rate:

$$\frac{f(x)}{\delta_D(x)} \le C \frac{f(y)}{\delta_D(y)}, \quad \text{for all } x, y \in B(Q, r/2) \cap D.$$

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Generalizations to various Lévy processes (Kim, Song, V) and to jump processes in metric spaces (Bogdan, Kumagai, Kwaśnicki).

Boundary Harnack principle, cont.

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The BHP for censored α-stable process, $\alpha \in (1,2)$: Let D be a $C^{1,1}$ open set in $\mathbb{R}^d$ and X a censored α-stable process. There exists $C > 0$ and $r_0 > 0$ such that for all $Q \in \partial D$, all $r \in (0, r_0)$ and all non-negative functions $f : D \to [0, \infty)$ that are harmonic (wrt X) in $D \cap B(Q, r)$ and vanish continuously on $B(Q, r) \cap \partial D$ it holds that

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Bogdan, Burdzy, Chen 2003

Carleson's estimate

An essential ingredient of most of the proofs of the BHP is Carleson's estimate: Let $D \subset \mathbb{R}^d$ be an open set. There exist constants $C > 0$ and $r_0 > 0$ such that for all $Q \in \partial D$, all $r \in (0, r_0)$ and all non-negative functions $f : \mathbb{R}^d \to [0, \infty)$ ($f : D \to [0, \infty)$) that are harmonic in $B(Q, r) \cap D$ and vanish in $B(Q, r) \cap D^c$ (vanish continuously on $B(Q, r) \cap \partial D$) it holds that

$$f(x) \le Cf(x_0), \quad \text{for all } x \in B(Q, r/2) \cap D,$$

where $x_0 \in B(Q, r) \cap D$ with $\delta_D(x_0) \ge r/2$.

Killed stable process

Let $Z = (Z_t, \mathbb{P}_x)$ be the isotropic α-stable process in $\mathbb{R}^d$, $\alpha \in (0,2)$, $(Q_t)_{t\geq 0}$ the corresponding semigroup: $Q_t f(x) := \mathbb{E}_x f(Z_t)$ $t \geq 0$, $f : \mathbb{R}^d \to \mathbb{R}$.

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For $D \subset \mathbb{R}^d$ open, let $\tau_D := \inf\{t > 0 : Z_t \notin D\}$, $Z_t^D := Z_t$ if $t < \tau_D$, ∂ (cemetary) otherwise, $Q_t^D f(x) := \mathbb{E}_x f(Z_t^D) = \mathbb{E}_x(f(Z_t), t < \tau_D)$ the corresponding semigroup.

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$$\mathcal{L}_1f:=\lim_{t\to 0}\frac{Q_t^Df-f}{t}$$

a possible definition of fractional Laplacian in D; usually called *fractional Laplacian in D with zero exterior condition:*. Notation: $-(-\Delta)^{\alpha/2}_{|D}$.

KSBM and SKBM

Let $W = (W_t, \mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$, $S = (S_t)_{t \geq 0}$ an independent $\alpha/2$-stable subordinator. Then W_{S_t} is a subordinate Brownian motion and $(Z_t) \stackrel{d}{=} (W_{S_t})$.

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$$\mathcal{L}_0 f=-(-\Delta_{|D})^{\alpha/2}f:=\frac{1}{|\Gamma(-\alpha/2)|}\int_0^\infty (P_t^D f-f)t^{-\alpha/2-1}dt$$

Another possible definition of a fractional Laplacian in D: *the fractional power of the Dirichlet Laplacian*.

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Another possible definition of a fractional Laplacian in D: *the fractional power of the Dirichlet Laplacian.* $\mathcal{L}_0\neq\mathcal{L}_1$

If $(\widetilde{Q}_t^D)_{t\geq 0}$ is the semigroup of Y^D, then $\widetilde{Q}_t^D f \leq Q_t^D f$, $f \geq 0$. Y^D is a "smaller" process then Z^D.

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Let S be a $\delta/2$-stable subordinator, $\delta \in (0,2]$, T a $\gamma/2$-stable subordinator, $\gamma \in (0,2)$, so that $(\delta/2)(\gamma/2) = \alpha/2$. Let $Z_t = W_{S_t}$ be a SBM, Z^D the KSBM (=killed δ-stable), and $Y_t^D = Z_{T_t}^D$ the subordinate killed Z.

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$$\mathcal{L} = -((-\Delta)_{|D}^{\delta/2})^{\gamma/2}.$$

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$$\mathcal{L} = -((-\Delta)^{\delta/2}_{|D})^{\gamma/2}.$$

Since $(\delta/2)(\gamma/2) = \alpha/2$, also a version of α-fractional Laplacian in D.

In two recent papers, Potential theory of subordinate killed Brownian motion *Trans.Amer. Math. Soc.* (2019) and On the boundary theory of subordinate killed Lévy processes, to appear in *Pot. Anal.* (2019), Kim, Song, V studied potential theory of operators like $\mathcal{L}$ (i.e. potential theory of Y^D) in order to understand how it depends on δ and γ.

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These subordinate processes have Dirichlet forms

$$\begin{aligned}\mathcal{E}(u,v) &= \frac{1}{2}\iint_{D\times D}(u(x)-v(x))(u(y)-v(y))J^D(x,y)\,dy\,dx \\ &\quad + \int_D u(x)v(x)\kappa(x)\,dx,\end{aligned}$$

where

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$$J^D(x,y) = c(\gamma)\int_0^\infty q^D(t,x,y)t^{-\gamma/2-1}\,dt, \quad x,y\in D,$$

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Here $q^D(t,x,y)$ are transition densities of Z^D (killed δ-stable).

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Define $B(x,y)$ by $J^D(x,y) = B(x,y)|x-y|^{-d-\alpha}$. We think of $B(x,y)$, $x,y \in D$, as a boundary part of the jumping kernel $J^D(x,y)$. Displays on the previous slide give sharp two-sided estimates of $B(x,y)$ for the subordinate killed process Y^D.

Boundary Harnack principle

Assume $\gamma \in (1,2)$ or $\delta = 2$. If D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$, then the boundary Harnack principle (with explicit decay rate) holds for Y^D:

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The non-scale-invariant boundary Harnack principle holds near the boundary of D if there is a constant $\widehat{R} \in (0,1)$ such that for any $r \in (0, \widehat{R}]$, there exists a constant $c = c(r) \geq 1$ such that for every $Q \in \partial D$ and any non-negative functions f, g in D which are harmonic in $D \cap B(Q,r)$ with respect to Y^D and vanish continuously on $\partial D \cap B(Q,r)$, we have

$$\frac{f(x)}{f(y)} \leq c\, \frac{g(x)}{g(y)} \quad \text{for all } x, y \in D \cap B(Q, r/2).$$

Assumptions on B

Let D be an open subset of $\mathbb{R}^d$ and $\alpha \in (0,2)$. Let $Y = Y^{D,\kappa}$ be a Hunt process with the Dirichlet form whose jumping kernel has the form

$$J^D(x,y) = B(x,y)|x-y|^{-\alpha-d}, \quad x,y \in D,$$

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(B1) $B(x,y) = B(y,x)$ for all $x,y \in D$.
(B2) There exists $C_2 > 0$ such that $B(x,y) \le C_2$.
(B3) For every $a \in (0,1)$ there exists $C_3 = C_3(a) > 0$ such that $B(x,y) \ge C_3$ whenever $\delta_D(x) \wedge \delta_D(y) \ge a|x-y|$.

Assumptions on B, cont.

(B4) There exists $\delta > 0$ and $C_4 > 0$ such that

$$0 \le 1 - B(x,y) \le C_4 \left(\frac{|x-y|}{\delta_D(x) \wedge \delta_D(y)} \right)^{\delta}$$

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(B5) For every $\epsilon \in (0,1)$ there exists $C_5 = C_5(\epsilon) \ge 1$ with the following property: For every $x_0 \in D$ and every $r > 0$ such that $B(x_0, (1+\epsilon)r) \subset D$, we have for all $x_1, x_2 \in B(x_0, r)$ and all $z \in D \setminus B(x_0, (1+\epsilon)r)$,

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(B6) There exist $\widehat{\beta} > 0$ and $C_7 > 0$ such that if $\delta_D(x) \leq \delta_D(z)$ and $|y-z| \leq M|y-x|$ with $M \geq 1$, then

$$B(x,y) \leq C_7 M^{\widehat{\beta}} B(z,y) .$$

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$$L(x,y) := \frac{\log\left(1 + \frac{(\delta_D(x) \vee \delta_D(y)) \wedge |x-y|}{\delta_D(x) \wedge \delta_D(y) \wedge |x-y|}\right)}{\log 2},$$

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Examples of B, cont.

All three boundary functions are comparable:

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Note that comparability implies that if the assumptions **(B2)**, **(B3)**, **(B5)** and **(B6)** hold for one of these functions, then they hold for the others as well.

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The boundary terms of subordinate killed stable processes satisfy assumptions **(B1)**-**(B6)**.

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Assume **(B1)**-**(B5)**. We first prove that for a sufficiently small open set $U \subset D$ which is sufficiently far away from the boundary ∂D, the Green function of Y killed upon exiting U, G_U^Y, is comparable to the Green function of the α-stable process when it exits U.

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Theorem (Harnack inequality):
(a) There exists a constant $C > 0$ such that for any $r \in (0, 1]$ and $B(x_0, r) \subset D$ and any Borel function f which is non-negative in D and harmonic in $B(x_0, r)$ with respect to Y, we have

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(b) For every $K > 0$ and every $L > 0$ there exists a constant $C = C(K, L) > 0$ such that for any $r < K/L$ and any Borel function f which is non-negative in D and harmonic in $B(x_1, r) \cup B(x_2, r)$ with respect to Y we have

$$f(x_2) \le CL^{d+\alpha} f(x_1).$$

Carleson's inequality

Theorem: Suppose that $D \subset \mathbb{R}^d$ is a κ-fat open set with characteristics $(R, \overline{\kappa})$ and assume that **(B1)-(B6)** hold true. There exists a constant $C = C(R, \overline{\kappa}) > 0$ such that for every $Q \in \partial D$, $0 < r < R/2$, and every non-negative function f in D that is harmonic in $D \cap B(Q, r)$ with respect to Y and vanishes continuously on $\partial D \cap B(Q, r)$, we have

$$f(x) \le Cf(x_0) \qquad \text{for } x \in D \cap B(Q, r/2),$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) \ge \overline{\kappa} r/2$.

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(B7) There exist a constant $C_8 \geq 1$ and parameters $\beta_1, \beta_2, \beta_3 \geq 0$ with $\beta_1 > 0$ if $\beta_3 > 0$ such that

$$C_8^{-1} \widetilde{B}(x, y) \leq B(x, y) \leq C_8 \widetilde{B}(x, y) \,, \quad x, y \in \mathbb{R}^d_+.$$

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Under **(B8)**, Y enjoys α-scaling: If $Y_t^{(r)} = rY_{r^{-\alpha}t}$, then $(Y^{(r)}, \mathbb{P}_{rx})$ has the same law as $(Y, \mathbb{P}_x)$.

The operator L^B

For $f:\mathbb{R}^d_+\to[0,\infty)$, we set for $x\in\mathbb{R}^d_+$,

$$L^B_\alpha f(x) := \text{p.v.}\int_{\mathbb{R}^d_+}\frac{f(y)-f(x)}{|y-x|^{d+\alpha}}B(x,y)\,dy = \text{p.v.}\int_{R^d_+}(f(y)-f(x))J^{\mathbb{R}^d_+}(x,y)\,dy\ ,$$

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The function $p\mapsto C(\alpha,p,B)$ is increasing and maps $((\alpha-1)_+,\alpha+\beta_1)$ onto $(0,\infty)$.

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so that $L^B g_p(x) = 0$.

Exit probabilities

For $a, b > 0$ define $D(a,b) := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : |\widetilde{x}| < a, 0 < x_d < b\}$ and $U(r) = D(\frac{r}{2}, \frac{r}{2})$. Write U for $U(1)$.

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The key to the proofs is finding a good testing function ϕ and then estimating $L^B \phi$.

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Recall **(B4)**: There exist $\delta > 0$ and $C_4 > 0$ such that

$$0 \le 1 - B(x,y) \le C_4 \left(\frac{|x-y|}{\delta_D(x) \wedge \delta_D(y)} \right)^{\delta}$$

whenever $\delta_D(x) \wedge \delta_D(y) \ge |x-y|$ and note that if $B = \widetilde{B}$ or $B = \widehat{B}$, then any $\delta > 0$ will do (in particular $\delta = 1$), while for $B = \bar{B}$, $\delta = \beta_1 + \beta_2$.

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Theorem (BHP): Suppose that $\delta > (\alpha - 1)_+$ and either
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In case (a)

$$\widetilde{B}(x,y) = \left(\frac{\delta_D(x)}{|x-y|} \wedge 1 \right)^{\beta} \left(\frac{\delta_D(y)}{|x-y|} \wedge 1 \right)^{\beta}$$

and $p \in ((\alpha - 1)_+, \alpha + \beta)$.

Failure of BHP

Theorem: Assume that $\alpha < p < \alpha + \beta_1$, $\alpha + \beta_2 < p$ and $\beta_3 \geq 0$. Then the non-scale-invariant boundary Harnack principle is not valid for $Y = Y^{\mathbb{R}^d_+, \kappa}$.