Persistence of AR(1)-sequences.

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(based on joint work with Günter Hinrichs and Martin Kolb)

Consider the iterations

$$X_{n+1} = aX_n + \xi_{n+1}, \quad n \geq 0,$$

where a is a positive constant, $\{\xi_n\}$ are i.i.d. random variables, and the starting value X_0 is either random or deterministic.

Define

$$T_0 := \inf\{n \geq 1 : X_n \leq 0\}.$$

We are interested in the tail properties of this stopping time.

If $a > 1$ then the sequence X_n grows exponentially fast with positive probability and, therefore, $T_0 = \infty$ with positive probability.

If $a = 1$ then X_n is a random walk.

If $a < 1$ then, under mild conditions on ξ_n, the chain X_n is positive recurrent:

$$X_n \Rightarrow \pi(dx), \quad \pi(A) := \mathbf{P}\left(\sum_{k=1}^{\infty} a^{k-1}\xi_k \in A\right).$$

Furthermore, $\mathbf{P}(T_0 < \infty) = 1$ provided that $\pi((-\infty, 0)) > 0$.

The recursion $X_{n+1} = aX_n + \xi_{n+1}$ looks quite contractive. So it is natural to expect that T has a geometrically decaying tail:

Theorem 1. Assume that

$$\mathbf{P}(\xi_1 > 0)\mathbf{P}(\xi_1 < 0) > 0, \quad \mathbf{E}\log(1 + |\xi_1|) < \infty \quad \text{and} \quad \mathbf{E}(\xi_1^+)^\delta < \infty.$$

Then,for every $a \in (0, 1)$,

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathbf{P}_x(T_0 > n) =: \lambda_a \in (0, \infty), \quad x > 0.$$

Furthermore, if the distribution of ξ_1 satisfies

$$\lim_{x\to\infty} \frac{\log \mathbf{P}(\xi_1 > x)}{\log x} = 0,$$

then

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathbf{P}_x(T_0 > n) = 0, \quad x > 0.$$

Remarks.

- The existence of the persistence exponent λ_a for $AR(p)$-sequences with absolutely continuous innovations has been shown by Aurzada, Mukherjee and Zeitouni, 2017.
- The second statement in Theorem 1 shows that the finiteness of $\mathbf{E}(\xi_1^+)^\delta$ for some $\delta > 0$ is the minimal condition for the positivity of λ_a: if $\mathbf{P}(\xi_1 > x)$ is slowly varying then $\mathbf{E}(\xi_1^+)^\delta = \infty$ for all $\delta > 0$ and $\lambda_a = 0$.

Explicit values of the persistence exponent.

- If $\mathbf{P}(\xi_k = 1) = 1 - \mathbf{P}(\xi_k = -1) = p \in (0,1)$ and $a \leq \frac{1}{2}$ then

$$\mathbf{P}_x(T_0 > n) = p^n.$$

- Aurzada, Mukherjee and Zetouni: If $a < 0$ and ξ_k are exponetially distributed then

$$\mathbf{P}_x(T_0 > n) = c(a)(1-a)^{-n}$$

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- Aurzada and Baumgarten, 2011: upper and lower bounds for λ_a in the case of normally distributed innovations.

Explicit expressions for the generating function of T_0.

Larralde, 2004, Novikov, 2009 and Christensen, 2012 have obtained explicit expressions for the generating function of T_0 in the case when the distribution of the innovations is related to the exponential distribution. These expressions are quite hard to analyse.

If, for example, $(1-a)-\xi_k$ are exponentially distributed then

$$\mathbf{E}_x s^{T_0} = \frac{sN_x(s)}{N_{1/a}(s)},$$

where

$$N_x(s) := \sum_{k=0}^{\infty} \frac{(ax)^k (s,a)_k}{k!}$$

and

$$(s,a)_k = \prod_{j=1}^{k} (1 - sa^{j-1}).$$

Standard compactness approach.

Assume that the distribution of the innovations is absolutely continuous and that the density φ is such that $\varphi(x) = 0$ for all $x > R$. Then the operator P_+ defined by

$$P_+ f(x) = \mathbf{E}[f(X_1); T_0 > 1] = \int_0^\infty f(y)\varphi(y - ax)dy$$

is compact on the space $C[0, R/(1-a)]$.

Then, by standard Perron-Frobenius-type arguments,

$$\mathbf{P}_x(T_0 > n) \sim V(x)e^{-\lambda_a n}.$$

If the innovations are not bounded from above, then, for every $n \geq 1$,

$$P_+^n \mathbf{1}(x) = \mathbf{P}_x(T_0 > n) \to 1, \quad x \to \infty.$$

Therefore,

$$\|P_+^n\| = 1 \quad \text{and} \quad r(P_+) = 1 > e^{-\lambda_a}.$$

This implies that P_+ is not compact on $C[0, \infty)$.

Similar can be shown that P_+ is not compact on standard L_p.

Baumgarten, 2013: P_+ is compact on $L_1(\phi(x)dx)$.

Alternative approach.

Fix $r > 0$ and define the stopping times

$$\sigma_r := \inf\{n \geq 1 : \ X_n > r\} \quad \text{and} \quad T_r := \inf\{n \geq 1 : \ X_n \leq r\}.$$

Then, for $\lambda \in \mathbb{C}$ with $|\lambda| < \lambda_a$,

$$\begin{aligned}\mathbf{E}_x[e^{\lambda T_0}] = \mathbf{E}_x\left[e^{\lambda T_0}; T_0 < \sigma_r\right] \\ + \mathbf{E}_x\left[e^{\lambda\sigma_r}\mathbb{I}\{T_0 > \sigma_r\}\mathbf{E}_{X_{\sigma_r}}[e^{\lambda T_0}; T_0 = T_r]\right] \\ + \mathbf{E}_x\left[e^{\lambda\sigma_r}\mathbb{I}\{T_0 > \sigma_r\}\mathbf{E}_{X_{\sigma_r}}[e^{\lambda T_0}; T_0 > T_r]\right].\end{aligned}$$

By the Markov property at time T_r,

$$\mathbf{E}_x[e^{\lambda T_0}] = F_\lambda(x) + \int_0^r K_\lambda(x, dy)\mathbf{E}_y[e^{\lambda T_0}].$$

Assume that the distribution of the innovations is absolutely continuous with a strict positive density φ, $\mathbf{E}(\xi_1^-)^\delta < \infty$ for some δ and $\mathbf{E}(\xi_1^+)^p < \infty$ for all p. Then there exists $\varepsilon > 0$ such that, for all $|\lambda| \le \lambda_a + \varepsilon$,

- F_λ is a continuous function,
- K_λ is compact on $C[0, r]$.

Then, using the Fredholm alternative, we conclude that $(I - K_\lambda)^{-1}$ is meromorphic. Therefore, $\mathbf{E}_x[e^{\lambda T_0}] = (I - K_\lambda)^{-1} F_\lambda(x)$ is meromorphic as well.

λ_a is a simple pole and there are no further poles on the circle $|\lambda| = \lambda_a$.

Theorem 2. Assume that the distribution of innovations has a density $\varphi(x)$ which is positive a.e. on $\mathbb{R}$. Assume also that $\mathbf{E}(\xi_1^+)^t < \infty$ for all $t > 0$ and $\mathbf{E}(\xi_1^-)^\delta < \infty$ for some $\delta > 0$. Then there exist $\gamma > 0$ and a positive function V such that

$$\mathbf{P}_x(T_0 = n) = e^{-\lambda_a(n+1)}V(x) + O\left(e^{-(\lambda_a+\gamma)n}\right).$$

The function V is e^{λ_a}-harmonic for the transition kernel P_+, that is,

$$V(x) = e^{\lambda_a}\int_0^\infty P_+(x, dy)V(y) = e^{\lambda_a}\mathbf{E}_x[V(X_1); T_0 > 1], \quad x \geq 0.$$

Remarks.

- Shumitzky and Wenska, 1975 have applied the Fredholm alternative to renewal-type equations appearing in branching processes.
- The existence of all moments of ξ_1^+ is the minimal condition for the purely geometric decay: If $\mathbf{P}(\xi_1 > x) = L(x)x^{-t}$ for some $t > 0$ and $L(x) = O(\log^{-2t-2} x)$ then
$$e^{\lambda_a n}\mathbf{P}_x(T_0 > n) \to 0.$$
(In this case one has $\lambda_a = -t \log a$.)
- The conclusion of Theorem 2 follows from Proposition 7.2 in Champagnat and Villemonais, 2017 provided that $\mathbf{E}e^{\log^{1+\delta} \xi_1^+}$ is finite.
- The operator P_+ is quasi-compact on the set of measurable functions equipped with the norm $\|f\|_\Lambda := \sup_{x\geq 0} |f(x)|/\Lambda(x)$, where $\Lambda(x) = \mathbf{E}_x[e^{(\lambda_a+\varepsilon)T_r}]$. Therefore, one can use the Perron-Frobenius arguments.